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TRANSCENDENTAL MEROMORPHIC SOLUTIONS OF SOME ALGEBRAIC DIFFERENTIAL EQUATIONS

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Abstract

In this paper we treat transcendental meromorphic solutions of some algebraic differential equations. We consider the number of distinct transcendental meromorphic solutions. Algebraic relations between meromorphic solutions and comparisons of the growth of transcendental meromorphic solutions are also discussed.


Keywords and phrases: Meromorphic function, Algebraic differential equation, Nevanlinna theory.

1. Introduction

The binomial differential equation

\[(y')^n = R(z, y),\]

where \(n\) is a positive integer and \(R(z, y)\) is a rational function in \(z\) and \(y\), has been studied under the assumption that it has a transcendental meromorphic solution \(y\) in the complex plane (for example, Yosida [18], Laine [10]). The result due to Steinmetz [14], Bank and Kaufman [1] states that by a suitable Möbius transformation \(v = (\alpha y + \beta)/(\gamma y + \delta), \alpha \delta - \beta \gamma \neq 0\), the binomial equation is classified into the following six simple differential equations:

\[
\begin{align*}
(v') &= a_2(z)v^2 + a_1(z)v + a_0(z) \\
(v')^2 &= a(z)(v - b(z))^2(v - \tau_1)(v - \tau_2) \\
(v')^3 &= a(z)(v - \tau_1)^2(v - \tau_2)(v - \tau_3)(v - \tau_4) \\
(v')^4 &= a(z)(v - \tau_1)^2(v - \tau_2)^2(v - \tau_3)^2 \\
(v')^5 &= a(z)(v - \tau_1)^2(v - \tau_2)^3(v - \tau_3)^3 \\
(v')^6 &= a(z)(v - \tau_1)^3(v - \tau_2)^4(v - \tau_3)^5
\end{align*}
\]
where $\tau_1, \ldots, \tau_4$ are distinct constants and $a_j(z)(\neq 0)$, $j = 0, 1, 2$, $a(z)$, $b(z)$ are rational functions. The result cited above of Steinmetz (Theorem 2 in [14]) was generalized to the case when $R(z, y)$ is rational in $y$ with meromorphic coefficients by v. Rieth [13] and He-Laine [9].

Throughout this paper ‘meromorphic’ means ‘meromorphic in the complex plane’ and we use the standard notation of the Nevanlinna theory of meromorphic functions (for example, [6], [11], [12]).

We consider the following three problems for the equations (II) especially when $b(z)$ is a constant, say (II$^*$), and (III). The equations (II$^*$) and (III) are treated in Section 2 and in Section 3 respectively.

The first problem is to classify the equations by the number of transcendental meromorphic solutions. The differential equations (I)–(VI) do not always admit transcendental meromorphic solutions. It depends on the coefficients of the equations. We investigate how many transcendental meromorphic solutions the differential equations have and under what conditions they have an infinite number of transcendental meromorphic solutions. We have some results for the Riccati equation (I) concerning numbers of meromorphic solutions, for example, [2], or [11] Chapter 9. Answers of this problem for (II$^*$) are given in Corollary to Theorem 2.2 (a), (b) and those for (III) are given in Corollary to Theorem 3.1 (a), (b).

The second problem is to find algebraic relations between meromorphic solutions. For the case of Riccati equation (I), four distinct solutions $f_1$, $f_2$, $f_3$, $f_4$ of (I) satisfy $\mathcal{R}(f_1, f_2, f_3, f_4) = c$, for a constant $c$, where $\mathcal{R}$ is a cross ratio of four elements (for example, [8], §4.2). We shall give an answer for (II$^*$) by showing Theorem 2.1, and give an answer for (III) by showing Theorem 3.1 (iii).

The third problem is to compare the growth of transcendental meromorphic solutions. There are many results on the growth of transcendental meromorphic solutions of these six differential equations (for example, [1], [14], [15]). The fact proved in [1] and in [15] is that for the transcendental meromorphic solutions $f$ of (II$^*$) or (III), the order of $f$ is a positive integral multiple of $1/2$, which is dependent on the coefficients of the equation. For example (cf. [15], Satz 1) for any solution of $(f')^2 = A(z)(f^2 - 1)$ the order of $f$ is equal to $1 + d/2$ when $d \geq -1$, where

$$A(z) = c_1z^d + c_2z^{d-1} + \cdots \quad \text{for } z \to \infty, \ c_1 \neq 0.$$ 

This says that for given (fixed) coefficients transcendental meromorphic solutions $f$ and $g$ of the equation have the same order of growth.

We shall give more detailed estimates of growth of transcendental meromorphic solutions of (II$^*$) by showing Theorem 2.1, and those of solutions of (III) by showing Corollary to Theorem 3.1 (c).
2. Results for the equation (II)

This section devotes to answer the problems, which we posed in Section 1, for the equation (II) when \( b(z) \) is a constant, say (II\(^*\)). The equation (II\(^*\)) is changed to

\[
(f')^2 = A(z)(f^2 - 1),
\]

where \( A(z) = (b - \tau_1)(b - \tau_2)a(z) \), by the linear transformation

\[
f = 2(\tau_2 - b)(v - \tau_1)/((\tau_2 - \tau_1)(v - b)) - 1.
\]

The equation (1) is more appropriate than (II\(^*\)) for us to investigate its solutions. We denote by \( S(A) \) the set of transcendental meromorphic solutions of (1) for a given rational function \( A \), and denote by \( \# S(A) \) the number of functions in \( S(A) \).

In this section we prove the following theorems and corollaries:

**Theorem 2.1.** Suppose that the differential equation (1) possesses distinct transcendental meromorphic solutions \( f \) and \( g \). Then there is a constant \( c \) such that

\[
f^2 + 2cfg + g^2 = 1 - c^2.
\]

Conversely, if there are two nonconstant meromorphic functions \( f \) and \( g \) satisfying (2), then the following relation holds:

\[
(f')^2/(f^2 - 1) = (g')^2/(g^2 - 1),
\]

so that if \( f \) is a solution of (1), so is \( g \).

**Corollary to Theorem 2.1.** Suppose that the differential equation (1) possesses transcendental meromorphic solutions \( f \) and \( g \). Then we have

\[
T(r, g) = T(r, f) + O(1).
\]

**Theorem 2.2.** Suppose that the differential equation (1) admits at least three transcendental meromorphic solutions. Then we have:

(i) There is a rational function \( \alpha(z) \) such that \( A(z) = \alpha(z)^2 \).

(ii) We can write \( \alpha(z) \) in (i) as a decomposition of partial fractions

\[
\alpha(z) = p(z) + \sum_{j=1}^{n} k_j(z - \tau_j)^{-1},
\]
where $p(z)$ is a polynomial not identically equal to 0, $k_j$ ($j = 1, \cdots, n$) are integers, and $\tau_j$ ($j = 1, \ldots, n$) are distinct constants. Moreover, for any transcendental meromorphic solution $f$, there exists a constant $C \in \mathbb{C}$ such that

$$f(z) = \cosh \left( \int_0^z p(z) \, dz + \sum_{j=1}^n \log(z - \tau_j)^{k_j} + C \right).$$

\textbf{Corollary to Theorem 2.2.} We have

(a) Suppose that the differential equation (1) admits at least three transcendental meromorphic solutions. Then $\# \mathcal{S}(A) = \infty$.

(b) For a rational function $A$, there are three possibilities on the number of transcendental meromorphic solutions of (1): $\# \mathcal{S}(A) = 0$, $\# \mathcal{S}(A) = 2$ or $\# \mathcal{S}(A) = \infty$.

We note that any nonconstant meromorphic solution $f$ of (1) satisfies the second order linear differential equation

$$f'' - \left( A'/2A \right) f' - Af = 0.$$  

In fact, differentiating (1), we have $2f'f'' = A'(f^2 - 1) + 2Af f'$. Combining this and (1), we obtain (7) since $f' \neq 0$.

For the proofs of Theorems 2.1 and 2.2, we need some lemmas given below.

\textbf{Lemma 2.3} ([4], Theorem 1). Let $F$ and $G$ be meromorphic functions. $F$ and $G$ satisfy $F^2 + G^2 = 1$ if and only if there is a meromorphic function $\beta(z)$ such that

$$F = 2\beta/(1 + \beta^2) \quad \text{and} \quad G = (1 - \beta^2)/(1 + \beta^2).$$

\textbf{Lemma 2.4.} Let $f$ be a nonconstant meromorphic function and put

$$R(z) = (f')^2/(f^2 - 1).$$

If $R(z)$ has poles, any pole of $R(z)$ is of order at most 2.

\textbf{Proof of Lemma 2.4} Given a pole $z_0$ of $R(z)$, $z_0$ is either a pole of $f$, a zero of $f(z) - 1$ or a zero of $f(z) + 1$. If $z_0$ is a pole of $f$, then a standard
pole order comparison of (8) implies that $R(z)$ has a double pole at $z_0$. By a similar reasoning, if $f(z) = \pm 1 + \sum_{j=k}^{\infty} \alpha_j(z - z_0)^j$ around $z_0$, then $R(z)$ is regular at $z_0$ when $k \geq 2$, while $R(z)$ has a simple pole at $z_0$ when $k = 1$.

**Lemma 2.5.** Suppose that a meromorphic function $\alpha$ is written in a neighborhood of $a_0$ as

$$\alpha(z) = k/(z - a_0) + h(z), \quad (k \neq 0),$$

where $h(z)$ is regular at $a_0$. Then, the differential equation

$$w'' - \left(\alpha'(z)/\alpha(z)\right)w' - \alpha^2(z)w = 0$$

has a single-valued meromorphic solution in a neighborhood of $a_0$ if and only if $k$ is equal to an integer.

**Proof of Lemma 2.5** From (9), it is easy to see that $a_0$ is a regular-singular point for (10) (see [7], Satz 3.2). The corresponding indicial equation at $a_0$ is

$$\rho(\rho - 1) + \rho - k^2 = \rho^2 - k^2 = 0$$

and its solutions are $\rho = k$ and $\rho = -k$. Therefore, it is easy to see that (10) has a nonconstant meromorphic solution in a neighborhood of $a_0$ if and only if $k$ is equal to an integer. \hfill \Box

**Proof of Theorem 2.1** Assume that $f$ and $g$ are transcendental meromorphic solutions to (1), namely

$$(f')^2 = A(f^2 - 1) \quad \text{and} \quad (g')^2 = A(g^2 - 1).$$

Further it follows from (7) that

$$(f'')^2 - (A'/2A)f' - Af = 0 \quad \text{and} \quad (g'')^2 - (A'/2A)g' - Ag = 0.$$  

Add two equations in (12), and then multiply the obtained equality by $2(f' + g')/A$ to obtain

$$\frac{2(f' + g')(f'' + g'')}{A} - \frac{A'}{A^2}(f' + g')^2 = 2(f + g)(f' + g'),$$

from which we have $((f' + g')^2/A)' = ((f + g)^2)'$ so that

$$(f' + g')^2/A = (f + g)^2 + c',$$

where $c'$ is a constant. From (11) and (13) we eliminate $A$, $f'$ and $g'$ to obtain (2), where $c = 1 + c'/2$. Next we suppose that two nonconstant
meromorphic functions $f$ and $g$ satisfy (2). When $c^2 = 1$, we have $f = \pm g$ and so the relation (3) holds. We consider the case $c^2 \neq 1$. Write (2) as
\[(14) \quad (f + cg)^2 + (1 - c^2)g^2 = 1 - c^2.\]
Differentiating the both sides of (14), we have
\[(15) \quad (f' + cg')(f + cg) + (1 - c^2)g'g = 0.\]
Combining (14) and (15), we obtain
\[(16) \quad (g'/g)^2/(1 - g^2) = (f'/f)^2/(1 - c^2).\]
Similarly we obtain by symmetry
\[(17) \quad (g'/g)^2/(1 - g^2) = (f'/f)^2/(1 - c^2).\]
We can write (15) as $f(f' + cg') = -g(g' + cf')$, so that the right-hand sides of (16) and (17) are equal, which results in the assertion. □

Proof of Corollary to Theorem 2.1 If $c^2 = 1$, then $f = \pm g$ and we have $T(r, f) = T(r, g)$. Hence we only treat the case $c^2 \neq 1$. From (2), we have
\[(f/g)^2 + 2cf/g + 1 = (1 - c^2)/g^2,\]
from which we have by Nevanlinna’s first fundamental theorem
\[2T(r, g) = 2T(r, f/g) + O(1).\]
Changing roles of $f$ and $g$, and using Nevanlinna’s first fundamental theorem we obtain the relation
\[2T(r, f) = 2T(r, g/f) + O(1) = 2T(r, f/g) + O(1).\]
Combining the two relations above, we obtain (4). □

Proof of Theorem 2.2 (i) By the hypothesis of this theorem and by Theorem 2.1, there are transcendental meromorphic functions $f$ and $g$ satisfying
\[f^2 + 2cfg + g^2 = 1 - c^2 \quad (c^2 \neq 1),\]
from which we have
\[f^2 + ((cf + g)/\sqrt{1 - c^2})^2 = 1.\]
By Lemma 2.3, there is a meromorphic function $\beta(z)$ such that
\[f = 2\beta/(1 + \beta^2).\]
We see that $\beta(z)$ is transcendental since so is $f$. Hence we see that

$$ A(z) = \frac{(f')^2}{f^2 - 1} = -\left( \frac{2\beta'}{1 + \beta^2} \right)^2 = \left( \frac{2i\beta'}{1 + \beta^2} \right)^2. $$

That is to say, $A(z) = \alpha(z)^2$ where $\alpha(z) = (2i\beta'(z))/(1 + \beta^2)$. Since $A(z)$ is a rational function, $\alpha(z)$ must be a rational function. □

**Proof of Theorem 2.2 (ii)** By Theorem 2.2 (i), we can write $A(z) = \alpha(z)^2$ for a rational function $\alpha(z)$. If $\alpha(z)$ has a pole, then the pole is simple by Lemma 2.4 and the residue at the pole must be an integer by Lemma 2.5. Hence we can write $\alpha(z)$ in the following form:

$$ \alpha(z) = p(z) + \sum_{j=1}^{n} k_j (z - \tau_j)^{-1}, $$

where $p(z)$ is a polynomial, $n$ is the number of poles of $\alpha(z)$, $k_j (j = 1, \ldots, n)$ are integers, and $\tau_j (j = 1, \ldots, n)$ are distinct constants. Put here $\zeta(z) = \int_{0}^{z} p(t)dt$. Then the meromorphic functions

(18)  \[ f_1(z) = e^{\zeta(z)} \prod_{j=1}^{n} (z - \tau_j)^{k_j} \quad \text{and} \quad f_2(z) = e^{-\zeta(z)} \prod_{j=1}^{n} (z - \tau_j)^{-k_j}, \]

which are linearly independent, satisfy the linear differential equation (10). Since any solution $f(z)$ of (1) solves (10), $f(z)$ is written by a linear combination of $f_1$ and $f_2$, say

(19)  \[ f(z) = C_1 f_1(z) + C_2 f_2(z), \]

where $C_1$ and $C_2$ are constants. As $f_1'(z) = \alpha(z)f_1(z)$, $f_2'(z) = -\alpha(z)f_2(z)$ and $f_1f_2 = 1$ from (18), by substituting (19) into (1), we obtain that $C_1C_2 = 1/4$. Therefore we see that for some $C \in \mathbb{C}$, $f(z)$ is represented in the form

$$ f(z) = \cosh \left( \zeta(z) + \sum_{j=1}^{n} \log(z - \tau_j)^{k_j} + C \right). $$

It is immediately concluded that if $p(z) \equiv 0$, then meromorphic solutions to (1) are rational functions, which is a contradiction. Hence $p(z) \not\equiv 0$, the assertion follows. □

**Proof of Corollary to Theorem 2.2 (a):** It follows from Proof of Theorem 2.2 (ii) if $A = \alpha^2$, where $\alpha$ satisfies (5), then a meromorphic function of the form (6) is a solution of (1). This implies that $\mathcal{S}(A)$ is an uncountable set when $p(z) \not\equiv 0$. This implies that if (1) possesses at least three distinct transcendental meromorphic solutions, then $\# \mathcal{S}(A) = \infty$.

(b): It is clear that if $f$ is a transcendental meromorphic solution of (1), then $-f$ is also a transcendental meromorphic solution of (1). By (a), $\# \mathcal{S}(A) \geq 3$ implies $\# \mathcal{S}(A) = \infty$. Therefore we have proved (b). □
Remark 2.6 We mention a condition which implies $\mathcal{S}(A)$ is an empty set: If $A$ has at least one pole of order not less than 3, then $\#\mathcal{S}(A) = 0$. This is a direct consequence of Lemma 2.4.

We showed (a) of Corollary to Theorem 2.2 by means of Theorem 2.2. We mention here that we get the same result by only using the algebraic relation (2) and the relation (4). In fact, by the hypothesis of this Corollary, there are two meromorphic functions $f$ and $g$ in $\mathcal{S}(A)$ satisfying

$$f^2 + 2c fg + g^2 = 1 - c^2 \quad (c^2 \neq 1),$$

from which we have

$$f^2 + ((cf + g)/\sqrt{1-c^2})^2 = 1.$$

This shows that $(cf + g)/\sqrt{1-c^2} \in \mathcal{S}(A)$ by (3) in Theorem 2.1 and (4) since $f \in \mathcal{S}(A)$. Put

$$h = (cf + g)/\sqrt{1-c^2} \quad \text{and} \quad F = \gamma f + \delta h,$$

where $\gamma$ and $\delta$ are constants satisfying $\gamma^2 + \delta^2 = 1$. Then

\begin{equation}
(20) \quad f^2 + h^2 = 1 \quad \text{and} \quad ff' = -hh'.
\end{equation}

Now, we are going to prove that $F \in \mathcal{S}(A)$. In fact, by (20)

\begin{equation}
(21) \quad (F')^2 = \gamma^2 (f')^2 + 2\gamma \delta f'h' + \delta^2 (h')^2
= \gamma^2 A(f^2 - 1) + \delta^2 A(h^2 - 1) - 2\gamma \delta f(f')^2 / h
= A(\gamma^2 f^2 + \delta^2 h^2 - 1) + 2\gamma \delta Afh
= A((\gamma f + \delta h)^2 - 1)
\end{equation}

since $(f')^2/h^2 = (h')^2/f^2 = -A$ by (20) and $f, h \in \mathcal{S}(A)$. It follows from (21) that $F = \gamma f + \delta h$ is a meromorphic solution of (1) and by (4), $\gamma f + \delta h \in \mathcal{S}(A)$. This proves the assertion.

3. Results for the equation (III)

In this section we are concerned with the differential equation of the type (III) in Section 1. It will be seen below that solutions of the equation (III) are closely connected with the Weierstrass $\wp$-function. We choose and fix a $\wp$-function satisfying

\begin{equation}
(22) \quad (\wp')^2 = 4\wp^3 - \tilde{g}_2 \wp - \tilde{g}_3,
\end{equation}

where $\tilde{g}_2$, $\tilde{g}_3$, are constants satisfying $27\tilde{g}_3^2 - \tilde{g}_2^3 \neq 0$. For the sake of brevity we put $G(x) = 4x^3 - \tilde{g}_2 x - \tilde{g}_3$, and we denote by $e_1, e_2, e_3$ the distinct roots
of $G(x) = 0$. For any solution $v$ of (III), we set

$$f(z) = \frac{\alpha}{v(z) - \tau_4} - \beta$$

with

$$\alpha = -\frac{(\tau_1 - \tau_4)(\tau_2 - \tau_4)(\tau_3 - \tau_4)}{4},$$

$$\beta = \frac{1}{12}(2\tau_1(\tau_1 + \tau_2 + \tau_3) - (\tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1) - 3\tau_4^2).$$

Then the equation of type (III) can be translated into the following form:

$$ (f')^2 = A(z)(4f^3 - \tilde{g}_2f - \tilde{g}_3) = A(z)G(f), $$

where $A(z) \neq 0$ is a rational function. We denote by $\mathfrak{T}(A)$ the set of transcendental meromorphic solutions of (23) for a given rational function $A$, and denote by $\#\mathfrak{T}(A)$ the number of functions in $\mathfrak{T}(A)$.

The purpose of this section is to show the following theorem and corollary:

**Theorem 3.1.** Suppose that the equation (23) admits two transcendental meromorphic solutions $f$ and $g$ such that $f \neq L(g)$ for some Möbius transformation $L$ such that $L(z) \neq z$. Then we have

(i) There exists a polynomial $a(z)$ such that $A(z) = a'(z)^2$.

(ii) Any $f(z) \in \mathfrak{T}(A)$ can be written by

$$ f(z) = \wp(a(z) + c), \quad c \in \mathbb{C}, $$

where $\wp$ is the Weierstrass $\wp$ function given in (22).

(iii) Let $u(z)$ and $v(z)$ denote arbitrary distinct transcendental meromorphic solutions of (23). Then there exists a constant $d_0 \in \mathbb{C}$, such that $U = u - d_0$ and $V = v - d_0$ satisfy an algebraic relation

$$ U^2V^2 - G_2UV - G_1(U + V) - G_0 = 0, $$

where $G_0$, $G_1$ and $G_2$ are constants.

Conversely if transcendental meromorphic functions $U$ and $V$ satisfy (25), then we have

$$ (U')^2/K(U) = (V')^2/K(V), $$

where $K(x)$ is a polynomial of degree 3, expressed as

$$ K(x) = 4x^3 + ((G_0 + G_2^2)/G_1)x^2 + 2G_2x + G_1. $$

**Corollary to Theorem 3.1.** We have
(a) If the equation (23) admits two transcendental meromorphic solutions \(f\) and \(g\) such that \(f \neq L(g)\) for some Möbius transformation \(L\) which is not identity, then \(#\Sigma(A) = \infty\).

(b) For a rational function \(A\), there are three possibilities on the number of \(\Sigma(A)\): \(#\Sigma(A) = 0\), \(#\Sigma(A) = 4\) or \(#\Sigma(A) = \infty\).

(c) For any transcendental meromorphic solutions \(f\) and \(g\) of (23), we have
\[
T(r, g) = T(r, f) + S(r),
\]
where \(S(r)\) is small with respect to \(T(r, f)\) and \(T(r, g)\).

We need the following results due to Bank and Kaufman [1, Lemma 5], and due to Valiron [16].

**Lemma 3.2.** Let \(H(w)\) be a polynomial having constant coefficients, and let \(w(z)\) be a nonconstant elliptic function of elliptic order \(q\), which is a solution of the differential equation \((w')^q = H(w)\). Then we have

(a) If \(c_0\) and \(c_1\) are complex numbers satisfying \(c_1^q = H(c_0)\), then there exists a complex number \(\zeta\) such that \(w(\zeta) = c_0\) and \(w'(\zeta) = c_1\).

(b) Any solution of the differential equation \((w')^q = H(w)\) which is meromorphic and nonconstant in a region of the plane must be of the form \(w(z + C)\) where \(C\) is a constant.

The lemma given below is also needed for Proof of Theorem 3.1.

**Lemma 3.3.** Suppose that (23) has distinct transcendental meromorphic solutions \(f\) and \(g\). If \(f\) and \(g\) have a common pole \(z_0\), then \(\varphi := f - g\) does not have a zero at \(z_0\).

**Proof of Lemma 3.3** We write \(A\) in a neighborhood of \(z_0\) as
\[
A(z) = R_A(z - z_0)^{\lambda} + O((z - z_0)^{\lambda+1}), \quad R_A \neq 0,
\]
where \(\lambda\) is an integer. Let \(\mu_f\) and \(\mu_g\) denote orders of poles of \(f\) and \(g\) at \(z_0\) respectively. From (23), \(-2(\mu_f + 1) = \lambda - 3\mu_f\), that is, \(\mu_f = 2 + \lambda\). Similarly we have \(\mu_g = 2 + \lambda\). For the sake of brevity we write \(\mu_f = \mu_g = \mu\).

Write \(f\) and \(g\) in a neighborhood of \(z_0\) as
\[
f(z) = \frac{R_f}{(z - z_0)^{\mu}} + O((z - z_0)^{-(\mu-1)}), \quad R_f \neq 0,
\]
\[
g(z) = \frac{R_g}{(z - z_0)^{\mu}} + O((z - z_0)^{-(\mu-1)}), \quad R_g \neq 0.
\]
Substituting these representations into (23) and comparing the coefficients of terms \((z - z_0)^{-2(\mu + 1)}\), we obtain

\[(32) \quad R_f = R_g = \mu^2/4R_A.\]

It follows from (23) that

\[(33) \quad (\varphi'/\varphi)(f' + g') = A(4(f^2 + fg + g^2) - \tilde{g}_2).\]

Assume that \(\varphi\) has a zero at \(z_0\) of order \(\sigma > 0\). We compare the coefficients of \((z - z_0)^{-2(\mu + 2)}\) in the Laurent expansions in both sides of (33). Using (32), we obtain

\[\sigma \left( -\frac{\mu^3}{4R_A} - \frac{\mu^3}{4R_A} \right) = R_A \left( 4\left( \frac{\mu^4}{16R_A^2} + \frac{\mu^4}{16R_A^2} + \frac{\mu^4}{16R_A^2} \right) \right),\]

that is, \(-\sigma = 3\mu/2\), which is absurd. We have thus proved Lemma 3.3. \(\square\)

Furthermore we mention a remark below to state some basic properties of solutions of (23).

**Remark 3.4** (A) Every solution \(f\) of (23) satisfies

\[(34) \quad f'' = \frac{A'(z)}{2A(z)} f' + \frac{A(z)}{2} (12f^2 - \tilde{g}_2).\]

Moreover, if \(f\) and \(g\) are distinct solutions of (23), then we have

\[(35) \quad \frac{\varphi''}{\varphi} - \frac{A'(z)}{2A(z)} \varphi' = 6A(z)(f + g) = 0, \quad \text{or} \quad (36) \quad \frac{\varphi''}{\varphi} - \frac{A'(z)}{2A(z)} \varphi' = 6A(z)(f + g),\]

where \(\varphi := f - g\).

(B) Let \(f\) be a transcendental meromorphic solution of (23). We introduce here the following four Möbius transformations:

\[
L_0(x) = x, \quad L_1(x) = \frac{e_1x + e_2 - e_1^2 - e_1e_2}{x - e_1},
\]

\[
L_2(x) = \frac{e_2x + e_3^2 - e_3 - e_2e_3}{x - e_2}, \quad L_3(x) = \frac{e_3x + e_1^2 - e_1 - e_3e_1}{x - e_3}.
\]

We see that \(L_j(f), \ j = 0, 1, 2, 3\) are also solutions of (23), which is verified by direct computations. Moreover, we assert that for any other Möbius transformation \(L(x) = (ax + b)/(cx + d), \Delta := ad - bc \neq 0\), the equation (23) is not solved by \(L(f)\). To show this, we assume that \(L(f)\) satisfies (23), that is,

\[(37) \quad \Delta^2 \left( \frac{(f')^2}{cf + d} \right) = A(z) \left( 4\left( \frac{af + b}{cf + d} \right)^2 - \tilde{g}_2 \left( \frac{af + b}{cf + d} \right) - \tilde{g}_3 \right).\]
First we treat the case $c = 0$. In this case we may assume that $d = 1$ and $a \neq 0$. Using (23) and (37), we eliminate $f'$ and obtain a polynomial in $f$ which must vanish. Then we have that $a = 1$ and $b = 0$ since $f$ is a transcendental function. This implies that $L$ must be $L_0$ in this case.

Next we consider the case $c \neq 0$. We may assume that $c = 1$ in this case. Using the same argument above, we obtain a polynomial in $f$ of degree 4 which must vanish. Since $f$ is transcendental, all coefficients must vanish. From the coefficients of $f^4$, $f^3$ and $f^2$, we obtain the following relations

\begin{align*}
(38) \quad 4a^3 - \tilde{g}_2 a - \tilde{g}_3 &= 0, \\
(39) \quad 12a^2 b - 4b^2 + 4a^3 d + 8abd - 4a^2 d^2 - b\tilde{g}_2 - 3ad\tilde{g}_2 - 4d\tilde{g}_3 &= 0,
\end{align*}

and

\begin{align*}
(40) \quad 4ab^2 + 4a^2 bd - bd\tilde{g}_2 - ad^2 \tilde{g}_2 - 2d^2 \tilde{g}_3 &= 0.
\end{align*}

From (38) and (40), we eliminate $\tilde{g}_3$. Then noting that $ad - b \neq 0$, we have

\begin{align*}
(41) \quad 4ab + d(8a^2 - \tilde{g}_2) &= 0.
\end{align*}

(i) When $a \neq 0$, substituting $b = -d(8a^2 - \tilde{g}_2)/(4a)$ (from (41)) and $\tilde{g}_3 = 4a^3 - \tilde{g}_2 a$ (from (38)) into (39), we obtain

\begin{align*}
(a + d)d(12a^2 - \tilde{g}_2)^2 &= 0.
\end{align*}

We note that $d(12a^2 - \tilde{g}_2) \neq 0$. In fact, if $d(12a^2 - \tilde{g}_2) = 0$, by (41) we obtain that $a = 0$ since $ad - b \neq 0$, which is a contradiction. We have

\begin{align*}
(42) \quad d &= -a
\end{align*}

and from (41) we have

\begin{align*}
(43) \quad b &= (8a^2 - \tilde{g}_2)/4.
\end{align*}

(ii) When $a = 0$, we have $\tilde{g}_3 = 0$ by (38) and $d\tilde{g}_2 = 0$ by (41). If $d \neq 0$, $\tilde{g}_2 = 0$. This implies that $27\tilde{g}_3^2 - \tilde{g}_2^3 = 0$, which is a contradiction. We have $d = 0$. Substituting $a = 0, d = 0$ into (39), we obtain the equality

\begin{align*}
b(4b + \tilde{g}_2) &= 0.
\end{align*}

As $b \neq 0$ in this case ($ad - b \neq 0$), we have $b = -\tilde{g}_2/4$.

(i) and (ii) imply that (42) and (43) hold in any case.

By (38), we see that $a$ coincides with one of the roots of $G(x) = 0$, say $e_1, e_2$ or $e_3$. We note that $\tilde{g}_2 = -4(e_1e_2 + e_2e_3 + e_3e_1)$ and $\tilde{g}_3 = 4e_1e_2e_3$. In view of (42) and (43) if $a = e_1$ then $b = e_1^2 - e_2^2 - e_1e_2$ and $d = -e_1$.

This implies that $L$ coincides with $L_1$. Similarly we see that $L = L_2$ when $a = e_2$, and $L = L_3$ when $a = e_3$. 

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Proof of Theorem 3.1 (i) Let \( f \) and \( g \) be two transcendental meromorphic solutions of (23) satisfying the hypothesis of this theorem. First we will show that \( A(z) \) in (23) has no poles. From (23),

\[
(44) \quad A(z) = (f')^2/G(f) = (g')^2/G(g) \quad \text{and} \quad G(f)/G(g) = (f'/g')^2.
\]

Suppose that \( A \) has a pole \( z_0 \). From (44), there are four possibilities:

(i.1) \( z_0 \) is a pole of \( f \) and a pole of \( g \),

(i.2) \( z_0 \) is a pole of \( f \) and a zero of \( G(g) \),

(i.3) \( z_0 \) is a pole of \( g \) and a zero of \( G(f) \),

(i.4) \( z_0 \) is a zero of \( G(f) \) and a zero of \( G(g) \).

Here we give a remark. In the cases (i.2)–(i.4) we consider the zero of \( G(f) \) and \( G(g) \). Assume that \( z_0 \) is a zero of \( G(f) \) and \( G(g) \). It gives that \( f \) has one of the \( e_j \) (\( j = 1, 2, 3 \)) point at \( z_0 \). Without loss of generality we may assume that it is an \( e_1 \) point. We set \( f_1 = L_1(f) \), where \( L_1 \) is given in Remark 3.4 (B), that is,

\[
(45) \quad f_1 = (e_1f + e_1^2 - e_2^2 - e_1e_2)/(f - e_1).
\]

Then we see by a simple computation that \( f_1 \) also satisfies (23) and \( z_0 \) is a pole of \( f_1 \). Hence the cases (i.2)–(i.4) reduce to the case (i.1), by using a suitable Möbius transformation which can be defined similar way to (45). Thus we have only to consider the case (i.1). Denote by \( \mu_A, \mu_f \) and \( \mu_g \) orders of pole \( z_0 \) for \( A, f \) and \( g \) respectively.

From (23), we have \( 2(\mu_f + 1) = 3\mu_f + \mu_A \), that is, \( 1 \leq \mu_f = 2 - \mu_A \). Hence \( \mu_f = \mu_A = 1 \), similarly \( \mu_g = 1 \). Here we consider the Laurent expansions of \( A, f \) and \( g \) in a neighborhood of \( z_0 \) as follows:

\[
A(z) = R_A/(z - z_0) + \alpha_A + O(z - z_0), \quad R_A \neq 0,
\]

\[
f(z) = R_f/(z - z_0) + \alpha_f + O(z - z_0), \quad R_f \neq 0,
\]

\[
g(z) = R_g/(z - z_0) + \alpha_g + O(z - z_0), \quad R_g \neq 0.
\]

From (32), \( R_f = R_g = 1/4R_A \). Further, substituting these representations into (23) and comparing the coefficients of terms \((z - z_0)^{-3}\), we have

\[
(46) \quad \alpha_f = \alpha_g = -R_f\alpha_A/3R_A = -\alpha_A/(12R_A^2)
\]

By the assumption of this lemma \( \varphi := f - g \) does not vanish identically and by (46) \( \varphi \) has a zero at \( z_0 \). However by Lemma 3.3 it is impossible that \( \varphi \) has a zero at \( z_0 \), a contradiction.

Secondly we will show that all zeros of \( A \) are of even order. Let \( z_1 \) be a zero of \( A \). From (44), if \( z_1 \) is a zero of \( f' \), (respectively \( g' \)) and if \( z_1 \) is not a zero of \( G(f) \), (respectively \( G(g) \)), then the order of zero of \( A \) at \( z_1 \) is an even integer. Hence we shall consider four possibilities:
Lemma 3.2, there exists $f$ by $D$ conditions.

However the right-hand side has a double pole, a contradiction.

Lemma 3.3, we obtain $1 \leq \lambda = \mu_f - 2 = \mu_g - 2$, which implies $\mu_f \geq 3$. Simply we write $\mu_f = \mu_g = \mu$.

Consider the Laurent expansions of $A, f$ and $g$ in a neighborhood of $z_1$. Denote by $R_A$ the coefficient of $(z - z_1)^{\mu - 2}$ in the expansion of $A$, and denote by $R_f, R_g$ the coefficients of $(z - z_1)^{-\mu}$ in the expansions of $f, g$ respectively.

From (23) similarly to (32), we have

\[(47) R_f = R_g = \mu^2/4R_A.\]

We see that the coefficient of the term $(z - z_1)^{-2}$ in the right-hand side of (36) is $6R_A(R_f + R_g) = 3\mu^2$ by (47).

We divide the behavior of $\varphi$ at $z = z_1$ into three cases, that is, $\varphi$ has a pole at $z_1$, $\varphi$ has a zero at $z_1$, or $\varphi$ does not have a pole nor a zero at $z_1$.

We first assume that $\varphi$ has a pole at $z_1$ of order $\nu$. Note that by (47) $\nu$ is at most $\mu - 1$. In the left-hand side of (36), the coefficient of double pole $z_1$ is $\nu(\nu + 1) + (\mu - 2)\nu/2 = \nu^2 + \nu\mu/2$. Hence we have $2\nu^2 + \nu\mu - 6\mu^2 = 0$, i.e., $\nu = -2\mu$ or $2\nu = 3\mu$. Since $\mu$ and $\nu$ are positive, $\nu = -2\mu$ is absurd. If $2\nu = 3\mu$, then we have that $\mu \leq -2$ using $\nu \leq \mu - 1$, which is also absurd.

Next we treat the case $\varphi$ has a zero at $z_1$. By the assumption, $\varphi := f - g$ does not vanish. Hence in view of Lemma 3.3 this case does not occur.

Finally we consider the case $\varphi$ does not have a pole nor a zero at $z_1$. In this case $z_1$ is a simple pole or a regular point of the left-hand side of (36). However the right-hand side has a double pole, a contradiction.

Therefore $A$ must be a polynomial whose zeros are of even order, which implies that there exists a polynomial $a$ such that $A = (a')^2$. $\square$

**Proof of Theorem 3.1** (ii) We follow the idea in the proofs of Lemma 3.2 (a) and (b), see Bank and Kaufman [1]. Let $f$ be a transcendental meromorphic solution of (23). We fix $z_0 \in \mathbb{C}$ which is not a pole of $f$ satisfying the conditions $a'(z_0) \neq 0, \varphi'(z_0) \neq 0$ and $f'(z_0) \neq 0$ (or $G(f(z_0)) \neq 0$). Denote by $D_0$ a fundamental parallelogram of $\varphi$ that contains $z_0$. Further we set $f(z_0) = b_0$ and $f'(z_0)/a'(z_0) = b_1$. Then from (23), $b_1^2 = G(b_0)$. In view of Lemma 3.2, there exists $z_1 \in D_0$ such that $\varphi(z_1) = b_0$ and $\varphi'(z_1) = b_1$. We set $\alpha(z) = a(z) + z_1 - a(z_0)$ and $f_1 = f_1(z) = \varphi(\alpha(z)) = \varphi(a(z) + z_1 - a(z_0))$. Then it holds $f_1'(z) = \varphi'(\alpha(z))a'(z) = \varphi'(\alpha(z))a'(z)$, and hence

\[ (f_1')^2 = (\varphi'(\alpha))^2(a')^2 = AG(\varphi(\alpha)) = AG(f_1) \]
which implies that \( f_1 \) is a meromorphic solution of (23). We have that

\[
\begin{align*}
(48) \quad f_1(z_0) &= \varphi(a(z_0) + z_1 - a(z_0)) = \varphi(z_1) = b_0 = f(z_0), \\
(49) \quad f_1'(z_0) &= \varphi'(a(z_0) + z_1 - a(z_0))a'(z_0) \\
&= \varphi'(z_1)a'(z_0) = b_1a'(z_0) = f'(z_0).
\end{align*}
\]

Set \( \psi = f - f_1 \). Then from (48) and (49) we have that \( \psi(z_0) = \psi'(z_0) = 0 \). We see that \( A'/2A \) and \( A \) is analytic at \( z_0 \) from our assumption. Regarding \( g \) as to \( f_1 \) and \( \varphi \) as to \( \psi \) in (25), we conclude that \( \psi = 0 \), that is, \( f \) and \( f_1 \) must coincide. This proves (ii). 

**Proof of Theorem 3.1** (iii) Let \( u \) and \( v \) denote meromorphic solutions of (23), and let \( a(z) \) be a polynomial given in (i). We may assume that \( u = u(z) = \varphi(a(z)) \) and we can write \( v = v(z) = \varphi(a(z) + c) \) for a constant \( c \in \mathbb{C} \) by (ii). Put \( \varphi(c) = d_0 \) and \( \varphi'(c) = d_1 \). Then by the addition formula of \( \varphi \)-function,

\[
\varphi(a(z) + c) = \frac{1}{4} \left( \frac{\varphi'(a(z)) - \varphi'(c)}{\varphi(a(z)) - \varphi(c)} \right)^2 - \varphi(a(z)) - \varphi(c),
\]

that is,

\[
(50) \quad v = \frac{1}{4} \left( \frac{\varphi'(a(z)) - d_1}{u - d_0} \right)^2 - u - d_0.
\]

Since \( d_1^2 = G(d_0) \) and \( (a'(z))^2 = A(z) \), from (23) and (50) we obtain

\[
(51) \quad \left( 4(u + d_0)(u - d_0)^2 - G(u) - G(d_0) \right)^2 = 4G(d_0)G(u).
\]

Put \( U = U(z) = u(z) - d_0 \) and \( V = V(z) = v(z) - d_0 \). Then since \( G(d_0) = 4d_0^3 - \tilde{g}_2d_0 - \tilde{g}_3 \) and \( G'(d_0) = 12d_0^2 - \tilde{g}_2 \), we can write (51) as

\[
U^2V^2 - \frac{1}{2} G'(d_0)UV - G(d_0)(U + V) + \frac{1}{16} (G'(d_0)^2 - 48d_0G(d_0)) = 0,
\]

which confirms that \( U \) and \( V \) satisfy a relation of the form (25).

Conversely we suppose that the relation (25) holds for meromorphic functions \( U \) and \( V \). We differentiate (25) to obtain

\[
(52) \quad U'(2UV^2 - G_2V - G_1) = -V'(2UV^2 - G_2U - G_1).
\]

Using (25) we have

\[
(53) \quad (2UV^2 - G_2U - G_1)^2 = 4U^2(G_2UV + G_1(U + V) + G_0) + G_2^2U^2 \\
+ G_1^2 - 4U^3VG_2 - 4U^2VG_1 + 2G_2G_1U \\
= 4G_1U^2 + (4G_0 + G_2^2)U^2 + 2G_1G_2U + G_1^2 \tag{15}
\]

\[
= G_1K(U).
\]
Similarly we obtain

\begin{equation}
(2UV^2 - G_2V - G_1)^2 = G_1K(V).
\end{equation}

Combining (52), (53) and (54), we obtain the assertion (26) with (27). □

**Proof of Corollary to Theorem 3.1** (a): We can see (a) from (ii) of Theorem 3.1.

(b): Suppose that (23) has a transcendental meromorphic solution \( f \). In the case there exists a transcendental meromorphic solution \( g \) of (23) such that \( g \neq L(f) \) for some Möbius transformation, we have that \( \#\Sigma(A) = \infty \) by (a). For the proof of (b) it remains to find the number of Möbius transformations \( L_j \) such that \( L_j(f) \) satisfy the equation (23) if \( \#\Sigma(A) \neq 0, \infty \). By means of Remark 3.4(B), the number of such Möbius transformations is equal to four, namely, \( \#\Sigma(A) = 4 \).

(c): In the case \( f = L(g) \) for a Möbius transformation \( L \), we have \( T(r,f) = T(r,g) + O(1) \) by means of the Nevanlinna first fundamental theorem. We may suppose that \( f \neq L(g) \) for any Möbius transformation \( L \). Then in view of Theorem 3.1(iii), for a \( d_0 \in \mathbb{C} \), \( f_0 = f - d_0 \) and \( g_0 = g - d_0 \) satisfy an algebraic relation (25). Since \( T(r,f_0) = T(r,f) + O(1) \) and \( T(r,g_0) = T(r,g) + O(1) \), it is enough to show that \( f_0 \) and \( g_0 \) satisfy the assertion of (c), namely \( T(r,f_0) = T(r,g_0) + O(1) \). If \( G_1 = 0 \) in (25), then \( f_0g_0 \) is a constant, from which we obtain that \( T(r,f_0) = T(r,g_0) + O(1) \). In what follows, we assume that \( G_1 \neq 0 \). Define meromorphic functions

\begin{equation}
(55) \quad f_1 = -(G_1g_0 + G_0)/f_0g_0^2 \quad \text{and} \quad g_1 = -(G_1f_0 + G_0)/g_0f_0^2.
\end{equation}

From (24) for \( U = f_0, V = g_0 \), we have

\[
f_0 - (G_2g_0 + G_1)/g_0^2 = (G_1g_0 + G_0)/f_0g_0^2.
\]

Eliminating \( f_0 \) of this equation by using the first one of (54), we see that \( f_1 \) and \( g_0 \) satisfy (25). Similarly we see that \( f_0 \) and \( g_1 \) satisfy (25). Namely,

\begin{align}
(56) \quad f_1^2g_0^2 - G_2f_1g_0 - G_1(f_1 + g_0) - G_0 &= 0, \\
(57) \quad f_0^2g_1^2 - G_2f_0g_1 - G_1(f_0 + g_1) - G_0 &= 0.
\end{align}

Thus \( f_0, g_0, f_1 \) and \( g_1 \) are transcendental meromorphic solutions of

\begin{equation}
(58) \quad (w')^2 = A(z)K(w),
\end{equation}

where \( A(z) \) is given in (23) and \( K(w) \) is given in (27). We also have

\begin{align}
(59) \quad f_0 + f_1 = (G_2g_0 + G_1)/g_0^2 \quad \text{and} \quad g_0 + g_1 = (G_2f_0 + G_1)/f_0^2.
\end{align}

It follows from (59) and \( G_1 \neq 0 \) that

\begin{equation}
(60) \quad 2T(r,g_0) \leq T(r,f_0) + T(r,f_1) + O(1).
\end{equation}
Using (55) and (59), we obtain
\[ f_0^{-1} + f_1^{-1} = -(G_{2g_0} + G_1)/(G_{1g_0} + G_0). \]

By means of the first fundamental theorem of Nevanlinna and (61),
\[ T(r, f) \leq T(r, g_0) + T(r, f_0) + O(1). \]

Combining (60) and (62), we have \( T(r, g_0) \leq 2T(r, f_0) + O(1) \). Changing the roles of \( f_0(z) \) and \( g_0(z) \), we obtain \( T(r, f_0) \leq 2T(r, g_0) + O(1) \). This implies that if \( \varphi(r) = S(r, f_0) \), then \( \varphi(r) = S(r, g_0) \), and if \( \varphi(r) = S(r, g_0) \), then \( \varphi(r) = S(r, f_0) \). Hence for two meromorphic functions \( f \) and \( g \), we can write \( S(r, f) = S(r) \) and \( S(r, g) = S(r) \).

We recall that some properties of a transcendental meromorphic solution \( w(z) \) of (58). Let \( w(z) \) be a transcendental meromorphic solution of (58). Then by means of Gol’dberg’s theorem [5], we see that \( w(z) \) is of finite order. We have that all poles of \( w(z) \) are double with a finite number of exceptions and \( m(r, w) = O(\log r) \). All zeros of \( w(z) \) are simple with a finite number of exceptions and \( m(r, 1/w) = O(\log r) \) since we assume \( G_1 \neq 0 \). Hence,
\[ N(r, w) = 2N(r, w) + O(\log r) = T(r, w) + O(\log r), \]
and
\[ N(r, 1/w) = N(r, 1/w) + O(\log r) = T(r, w) + O(\log r). \]

Let \( z_0 \) be a pole of \( f_0(z) \), and let \( z_1 \) be a pole of \( f_1(z) \). Then we see from (25) and (56) (or (55)), \( z_0 \) is a zero of \( g_0 \), and \( z_1 \) is also a zero of \( g_0 \). If both \( f_0(z) \) and \( f_1(z) \) have a common double pole \( z_2 \), then \( z_2 \) is a zero of \( g_0(z) \) of multiplicity at least two. From (64), the counting function of such common poles is of \( O(\log r) \). Thus it concludes that
\[ N(r, f_0) + N(r, f_1) \leq N(r, 1/g_0) + O(\log r). \]

From (63), (64) and (65),
\[ T(r, f_0) + T(r, f_1) \leq 2T(r, g_0) + O(\log r). \]

Combining this and (60), we obtain
\[ T(r, f_0) + T(r, f_1) = 2T(r, g_0) + O(\log r). \]

Further we define
\[ g_2 = -(G_1f_1 + G_0)/g_0f_1^2 \quad \text{and} \quad f_2 = -(G_1g_1 + G_0)/f_0g_1^2. \]

Repeating this process, we define sequences of meromorphic functions \( f_0, f_1, g_2, g_3, \ldots, g_0, f_1, g_2, f_3, \ldots \). Namely we set for \( k = 0, 1, 2, \ldots \),
\[ f_{2k+3} = -\frac{G_1g_{2k+2} + G_0}{f_{2k+1}g_{2k+2}}^2, \quad g_{2k+2} = -\frac{G_1f_{2k+1} + G_0}{g_{2k}f_{2k+1}^2}^2. \]
Assume that (69) and (70) hold for (71)

In view of (66) and the comment that we posed after the definitions of (70)

and we assert that there exist sequences (68) and (69) that satisfy (25) and that all triples

satisfy (66). We write for (67)

Let \(a_0, b_0, a_1\) and \(b_1\) be positive constants. We assume that there exists a sequence \(\{r_n\}\), \(r_n \to \infty\) as \(n \to \infty\) satisfying

Then we see that all functions

\[ \{f_j(z)\} \quad (j = 0, 1, \ldots) \quad \text{and} \quad \{g_k(z)\} \quad (k = 0, 1, \ldots) \]

are transcendental and satisfy the differential equation (58), all pairs

\[ (f_j(z), g_{j+1}(z)) \quad \text{and} \quad (g_j(z), f_{j+1}(z)) \quad (j = 0, 1, \ldots) \]

satisfy (25) and that all triples

\[ (f_{j-1}(z), f_j(z), g_j(z)) \quad \text{and} \quad (g_{j-1}(z), g_j(z), f_j(z)) \quad (j = 1, 2, \ldots) \]

satisfy (66). We write for \(j = 0, 1, 2, \ldots,\)

\[ h_j(z) = \begin{cases} f_j(z), & \text{if } j \text{ is odd} \\ g_j(z), & \text{if } j \text{ is even.} \end{cases} \]

Let \(a_0, b_0, a_1\) and \(b_1\) be positive constants. We assume that there exists a sequence \(\{r_n\}\), \(r_n \to \infty\) as \(n \to \infty\) satisfying

\[ \begin{cases} T(r_n, h_0) & \leq a_0 T(r_n, f_0) + O(\log r_n) \\ T(r_n, h_0) & \geq b_0 T(r_n, f_0) + O(\log r_n) \end{cases} \tag{67} \]

and

\[ \begin{cases} T(r_n, h_1) & \leq a_1 T(r_n, f_0) + O(\log r) \\ T(r_n, h_1) & \geq b_1 T(r_n, f_0) + O(\log r) \end{cases} \tag{68} \]

We assert that there exist sequences \(\{a_j\}\) and \(\{b_j\}\), \(j = 0, 1, 2, \ldots,\), such that

\[ T(r_n, h_j) \leq a_j T(r_n, f_0) + O(\log r_n) \tag{69} \]

and

\[ T(r_n, h_j) \geq b_j T(r_n, f_0) + O(\log r_n) \tag{70} \]

In view of (66) and the comment that we posed after the definitions of \(\{f_j(z)\}\) and \(\{g_j(z)\}\), we have for \(j = 1, 2, \ldots,\)

\[ T(r_n, h_{j-1}) + T(r_n, h_{j+1}) = 2T(r_n, h_j) + O(\log r_n). \tag{71} \]

Assume that (69) and (70) hold for \(j = 0, 1, 2, \ldots, k\). Then from (71),

\[ T(r_n, h_{k+1}) = 2T(r_n, h_k) - T(r_n, h_{k-1}) + O(\log r_n) \leq 2a_k T(r_n, f_0) - b_{k-1} T(r_n, f_0) + O(\log r_n), \]
which gives

\[ a_{k+1} = 2a_k - b_{k-1}. \]  

Similarly, we obtain

\[ b_{k+1} = 2b_k - a_{k-1}. \]  

Therefore, using the assumptions (67) and (68), we obtain \( \{a_n\} \) which satisfies (69) and \( \{b_n\} \) which satisfies (70) recursively by (72) and (73).

We now compute \( a_k \) and \( b_k \) concretely. Put \( c_k = a_k + b_k \). Then we have that \( c_{k+1} - 2c_k + c_{k-1} = 0 \), and hence \( c_k = (c_1 - c_0)k + c_0 \), \( k = 0, 1, 2, \ldots \). Thus we obtain

\[ c_k = \left( c_1 - c_0 \right)k + c_0, \]

where \( \mu = c_0 - c_1 \) and \( \nu = c_1 - 2c_0 \). In (74), we set \( d_k = a_{k+1} - a_k \). Then we have

\[ d_k - 2d_{k-1} - d_{k-2} = \mu. \]

Further, we put \( e_k = d_k + \mu/2 \). Then

\[ e_k - 2e_{k-1} - e_{k-2} = 0. \]  

Thus we can write \( e_k \) with some constants \( \gamma_1 \) and \( \gamma_2 \):

\[ e_k = \gamma_1 \lambda_1^k + \gamma_2 \lambda_2^k, \]

where \( \lambda_1 = 1 + \sqrt{2} \) and \( \lambda_2 = 1 - \sqrt{2} \), (roots of the equation \( t^2 - 2t - 1 = 0 \)), see for example [3]. Thus \( d_k = e_k - \mu/2 \), and hence for \( k = 1, 2, \ldots \),

\[ a_k = \sum_{j=0}^{k-1} d_j + a_0 = \sum_{j=0}^{k-1} \left( e_j - \frac{\mu}{2} \right) + a_0 = \sum_{j=0}^{k-1} \left( \gamma_1 \lambda_1^j + \gamma_2 \lambda_2^j - \frac{\mu}{2} \right) + a_0 \]

\[ = \gamma_1 \frac{1 - \lambda_1^k}{1 - \lambda_1} + \gamma_2 \frac{1 - \lambda_2^k}{1 - \lambda_2} - \frac{\mu}{2}k + a_0, \]

and

\[ b_k = 2a_k - a_{k+1} = \frac{\gamma_1}{1 - \lambda_1}(1 - 2\lambda_1^k + \lambda_1^{k+1}) + \frac{\gamma_2}{1 - \lambda_2}(1 - 2\lambda_2^k + \lambda_2^{k+1}) - \frac{\mu}{2}(k - 1) + a_0. \]

We assert that

\[ \liminf_{r \to \infty} \frac{T(r, f_1)}{T(r, f_0)} \geq 1 \quad \text{and} \quad \liminf_{r \to \infty} \frac{T(r, g_1)}{T(r, g_0)} \geq 1. \]
To show this, we assume that
\begin{equation}
\liminf_{r \to \infty} T(r, f_1)/T(r, f_0) = \alpha < 1.
\end{equation}

For any \( \epsilon > 0 \) such that \( \alpha + \epsilon < 1 \), there exists a sequence \( \{r_n\} = \{r_n(\epsilon)\} \) satisfying
\begin{equation}
T(r_n, f_1) \leq (\alpha + \epsilon)T(r_n, f_0) \quad \text{and} \quad T(r_n, f_1) \geq (\alpha - \epsilon)T(r_n, f_0),
\end{equation}
for \( n \geq n_0(\epsilon) \). Later we choose a suitable \( \epsilon \). From \( 66 \),
\begin{align*}
T(r_n, h_0) &= T(r_n, g_0) = \left( T(r_n, f_0) + T(r_n, f_1) \right) /2 + O(\log r_n) \\
&\leq \left( T(r_n, f_0) + (\alpha + \epsilon)T(r_n, f_0) \right) /2 + O(\log r_n) \\
&= \left( 1 + \alpha + \epsilon \right) T(r_n, f_0) /2 + O(\log r_n).
\end{align*}
Similarly, we have
\begin{align*}
T(r_n, h_0) &\geq \left( 1 + \alpha - \epsilon \right) T(r_n, f_0) /2 + O(\log r_n).
\end{align*}

We now set
\begin{equation*}
a_0 = (1 + \alpha + \epsilon)/2, \quad b_0 = (1 + \alpha - \epsilon)/2, \quad a_1 = \alpha + \epsilon, \quad \text{and} \quad b_1 = \alpha - \epsilon.
\end{equation*}

We compute \( \mu, \nu, \gamma_1 \) and \( \gamma_2 \) concretely under our assumptions. We have \( \mu = c_0 - c_1 = (a_0 + b_0) - (a_1 + b_1) = 1 - \alpha \), and \( \nu = c_1 - 2c_0 = -2 \). From \( 74 \),
\begin{equation*}
a_2 = 2a_1 + a_0 + \mu + \nu = (3/2)\alpha + (5/2)\epsilon - 1/2.
\end{equation*}

On the other hand, from \( 77 \),
\begin{align*}
a_1 &= \gamma_1 + \gamma_2 + \alpha + \epsilon/2 \\
a_2 &= (1 + \lambda_1)\gamma_1 + (1 + \lambda_2)\gamma_2 + (3/2)\alpha + (1/2)\epsilon - 1/2.
\end{align*}

Hence we have
\begin{align*}
\begin{cases}
\gamma_1 + \gamma_2 = \epsilon/2 \\
(1 + \lambda_1)\gamma_1 + (1 + \lambda_2)\gamma_2 = 2\epsilon.
\end{cases}
\end{align*}

Since \( \lambda_1 = 1 + \sqrt{2} \) and \( \lambda_2 = 1 - \sqrt{2} \), we obtain
\begin{equation*}
\gamma_1 = ((1 + \sqrt{2})/4)\epsilon, \quad \gamma_2 = ((1 - \sqrt{2})/4)\epsilon.
\end{equation*}

Hence we can write
\begin{equation}
a_k = (\alpha - 1)k + (1 + \alpha + \epsilon)/2 \\
+ \epsilon \left( \left( \frac{1 + \sqrt{2}}{4} \right) \left( \frac{1 + \sqrt{2}}{2} \right)^k - 1 \right) + \left( \frac{1 - \sqrt{2}}{4} \right) \left( \frac{1 - (1 - \sqrt{2})^k}{\sqrt{2}} \right).
\end{equation}
Since we assume that $\alpha < 1$, we can take $k = k(\alpha)$ so large that $(\alpha - 1)k + 1 < 0$. Once we find a such $k$, we fix it. Then we choose $\epsilon$ so small that $a_k < 0$. For this $\epsilon$, there exists $\{r_n\} = \{r_n(\epsilon)\}$ satisfying (80), in particular,

\begin{equation}
T(r_n, h_j) \leq a_k T(r_n, f_0) + O(\log r_n).
\end{equation}

We observe the term $O(\log r_n)$ in (82). Write this term $\psi(\log r_n)$. Then function $\psi(x)$ in $x$ depends on $k$. However, it is independent of $\epsilon$. Since $h_0$ is transcendental and $a_k < 0$, the right hand side of (82) is negative for sufficiently large $n$, a contradiction. This gives the first inequality in (78).

On the other hand, we consider a sequence of functions $h^*_j(z) = \begin{cases} f_j(z), & \text{if } j \text{ is even} \\ g_j(z), & \text{if } j \text{ is odd} \end{cases}$ instead of $h_j(z)$ above. Then we obtain the second inequality in (78) by similar arguments. Hence the assertion (78) follows. It follows from (66) and the first inequality in (78) that

\begin{equation}
\liminf_{r \to \infty} T(r, g_0)/T(r, f_0) \geq 1.
\end{equation}

We recall the remark that we posed after the definitions of $\{f_j(z)\}$ and $\{g_j(z)\}$, in particular,

\begin{equation}
T(r, g_0) + T(r, g_1) = 2T(r, f_0) + O(\log r).
\end{equation}

From this and the second inequality in (78), we have

$$
\liminf_{r \to \infty} T(r, f_0)/T(r, g_0) \geq 1,
$$

and hence

\begin{equation}
\limsup_{r \to \infty} T(r, g_0)/T(r, f_0) \leq 1.
\end{equation}

Hence we see that $\lim_{r \to \infty} T(r, f_0)/T(r, g_0) = 1$. This implies that $T(r, f_0) = (1 + o(1))T(r, g_0)$ as $r \to \infty$, which gives (28). We finally comment that the case $h_j(z) = h_i(z)$ for some $j \neq i$ is included in our arguments. We have thus proved (c).

\[\square\]

4. Examples

Finally we state some examples in this section. As mentioned in the statement in Theorem 2.2, a condition that gives $\#S(A) \geq 3$ is obtained and in Remark 2.6, a condition that gives $\#S(A) = 0$ is obtained.
A natural question arises: Under what conditions does \( \#S(A) = 2 \) occur? We shall give some examples of \( A \) in (1) for which \( \#S(A) = 2 \) and an example for (23) having the property \( \#T(A) = 4 \).

**Example 4.1** \( S(1/4z) = \{ \cosh \sqrt{z}, - \cosh \sqrt{z} \} \).

In fact, it is easy to see that \( \cosh \sqrt{z} \) and \( - \cosh \sqrt{z} \) are transcendental entire solutions of the differential equation

\[
(f')^2 = \frac{1}{(4z)}(f^2 - 1).
\]

It follows from Theorem 2.2 (i) that there is no other solution to the equation above.

Similarly let \( p(z) \) be a polynomial with simple zeros only. Then,

\[
\begin{align*}
S(p(p')^2) &= \{ \pm \cosh((2/3)p^{3/2}) \}, \\
S((p')^2/p) &= \{ \pm \cosh 2p^{1/2} \}, \\
S(z^2 - 1) &= \{ \pm \cosh(z\sqrt{z^2 - 1} - \log(z + \sqrt{z^2 - 1})) \}
\end{align*}
\]

**Example 4.2** The equation

\[
(f')^2 = \frac{1}{(4z)}(4f^3 - \tilde{g}_2f - \tilde{g}_3)
\]

possesses a solution \( \wp(\sqrt{z}) \), where \( \wp \) is Weierstrass' elliptic function satisfying (22). Clearly \( \wp(\sqrt{z} + c), c \neq 0 \in \mathbb{C} \) is not meromorphic and hence \( \#T(1/4z) = 4 \).

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