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New Approximate Formulae for Hopf's Function

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Hopf関数の新しい近似式

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要　旨

Hopf関数は，放射平衡にある平面平行近似の灰色大気の源泉関数において光学的深さの非線形部分を表わす関数である．灰色大気の源泉関数は，モデル大気を求める逐次計算において第1近似の源泉関数としてよく用いられる他，放射強度や放射流束の数値計算の精度を見積る際にもよく用いられる．そのような場合，Hopf関数の値を精度よく短い時間で計算しなければならない．

そのためのHopf関数の近似式として，Kourganoffが導いた6次までの積分指数関数による展開式およびそれにΛ積分を施した展開式が利用されてきた．最近，篠塚が，Hopf関数の近似式として8次までの積分指数関数による展開式およびそれにΛ積分を施した展開式をKourganoffと同じ方法で導いた．その結果，Kourganoffの近似式が$\pm 3\times10^{-4}\%$の精度なのに対し，篠塚の近似式は$\pm 2\times10^{-5}\%$の精度が得られ，10倍以上精度が上がった．

本研究では，篠塚の近似式の係数を平均誤差が最小になるように改良した．その結果，$\tau \geqq 0.01$の範囲では$\pm 4\times10^{-6}\%$の精度で値が得られることになり，篠塚の近似式の3倍以上，Kourganoffの近似式の60倍以上精度が上がった．なお，Kourganoffの近似式に比べ，本研究および篠塚の近似式では展開式の次数が増した分，計算時間が増すが，その割合は高々25%にすぎない．

I. Introduction

Hopf's function is the nonlinear part of the source function for the gray and plane-parallel atmosphere which is in radiative equilibrium. It is defined by

$$S(\tau)=\frac{3}{4}F\,[\tau+q(\tau)], \tag{1}$$

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where τ is the optical depth, F is the integrated astrophysical flux and $q(\tau)$ is Hopf's function ; $S(\tau)$ is the source function for the gray and plane-parallel atmosphere in radiative equilibrium.

Accurate values of Hopf's function for several tens of τ values have been calculated. For example, the $q(0)$ value has been derived theoretically to be $1/\sqrt{3}$ (Chandrasekhar (1960)[1]), and $q(\infty)$ value was calculated to be 0.71044609 by Placzek and Seidel (1947)[2]. Kourganoff (1952)[3] gave a table of the $q(\tau)$ values obtained by the iterated variational method, and Mihalas (1978)[4] gave one obtained by numerical integration. The table by Kourganoff (1952)[3] gives the $q(\tau)$ values for 30 τ values, and the table by Mihalas (1978)[4] gives the $q(\tau)$ values for 20 τ values. These tables give the $q(\tau)$ values with 6 significant figures. However, there are several discrepancies in the $q(\tau)$ values between these tables. Yoshioka (1985)[5] gave a table of the $q(\tau)$ values for 70 τ values obtained by numerical integration. This table gives the $q(\tau)$ values with 9 significant figures. Furthermore, Yoshioka (1985)[5] calculated by numerical integration the $q(\infty)$ value to be 0.710446089599.

The source function $S(\tau)$ is often used as the first approximation to the source function in iterative calculations of the temperature distribution in model atmospheres. It is also used for the evaluation of the accuracy of quadrature formulae for the mean intensity and the flux integrals.

Approximate formulae for Hopf's function have been used for model atmospheres and for this evaluation. The approximate formulae with high accuracy are necessary especially for the evaluation of the quadrature formulae with high accuracy. Many approximate formulae for Hopf's function have been proposed. Among the formulae widely known, the most accurate one is that obtained from the sixth order lambda iterated variational method by Kourganoff (1952)[3]. According to the estimation by Kourganoff (1952)[3], this formula gives the $q(\tau)$ values which are correct to within $\pm$ 0.000002 ($\pm$0.0003%).

Recently, Shinozuka (1991)[6] obtained the approximate formula from the eighth order lambda iterated variational method. In this paper, the coefficients of the approximate formula by Shinozuka (1991)[6] have been modified and the accuracy of this formula has been improved.

II. The Approximate Formula by Kourganoff

II.1. The Principle of the Lambda Iterated Variational Method

Kourganoff (1952)[3] expressed Hopf's function $q(\tau)$ approximately by an expansion of the following form,

$$q(\tau) \fallingdotseq A_0 + \sum_{j=2}^{n} A_j E_j(\tau). \tag{2}$$

In this expansion, $E_j(\tau)$ is the exponential integral function of order j, which is defined by

$$E_j(t)=\int_1^\infty \frac{e^{-tx}}{x^j}\,dx. \tag{3}$$

Kourganoff (1952)[3] determined the coefficients A_0, A_2, A_3, ……, A_n by the following variatinal method.

Under the condition of radiative equilibrium, $S(\tau)$ must satisfy the following relation,

$$\Phi\tau\{S(t)\}=F, \tag{4}$$

where the operator $\Phi\tau$ is the integral operator which transforms a source function into an astrophysical flux and it is defined by

$$\Phi\tau\{S(t)\}\equiv 2\int_0^\infty S(\tau+t)E_2(t)\,dt-2\int_0^\tau S(\tau-t)E_2(t)\,dt. \tag{5}$$

By substituting from the definition (1) into the relation (4), we get the following relation,

$$\Phi\tau\ \{\frac{3}{4}F[t+q(t)]\}=F, \tag{6}$$

By substituting from the expansion (2) into the relation (6), we get the following approximate relation,

$$A_0\phi_0(\tau)+\sum_{j=2}^{n}A_j\phi_j(\tau)\fallingdotseq\frac{4}{3}-\phi_1(\tau), \tag{7}$$

where

$$\phi_0(\tau)=\Phi\tau\{1\}=2\,E_3(\tau), \tag{8}$$

$$\phi_1(\tau)=\Phi\tau\{t\}=\frac{4}{3}-2\,E_4(\tau), \tag{9}$$

and

$$\phi_j(\tau)=\Phi\tau\{E_j(t)\}(j\geqq 2). \tag{10}$$

In the relation (7), Kourganoff (1952)[3] gave to τ a sequence of m discrete values τ_1, τ_2, ……, τ_m and solved, by the method of least squares, the system of m equations of condition,

$$A_0\phi_0(\tau_k)+\sum_{j=2}^{n}A_j\phi_j(\tau_k)=\frac{4}{3}-\phi_1(\tau_k), \tag{11}$$

where k=1, 2, ……, m. On writing

$$\omega_{pq}=\sum_{k=1}^{m}\phi_p(\tau_k)\phi_q(\tau_k), \tag{12}$$

and

$$\omega_q=\sum_{k=1}^{m}\phi_q(\tau_k), \tag{13}$$

we get the following system of normal equations which corresponds to the system of the equations (11),

$$\omega_{0q}A_0+\sum_{j=2}^{n}\omega_{jq}A_j=\frac{4}{3}\omega_q-\omega_{1q}. \tag{14}$$

With the solution $A_1, A_2, \cdots\cdots, A_m$ of the system of the equations (14), the expansion (2) gives an approximate formula for Hopf's function.

Moreover, Kourganoff (1952)[3] applied the operator $\Lambda\tau$ to the source function $S(\tau)$ where $q(\tau)$ is approximated by the expansion (2). The operator $\Lambda\tau$ is the integral operator which transforms a source function into a mean intensity and it is defined by

$$\Lambda\tau\{S(t)\} \equiv \frac{1}{2}\int_0^\infty S(\tau+t)E_1(t)\,dt + \frac{1}{2}\int_0^\tau S(\tau-t)E_1(t)\,dt. \tag{15}$$

By substituting from the expansion (2) into the definition (15), we get the following expression,

$$\Lambda\tau\{S(t)\} \fallingdotseq \frac{3}{4}F\{\tau+\frac{1}{2}E_3(\tau)+A_0\lambda_0(\tau) + \sum_{j=2}^{n} A_j\lambda_j(\tau)\}, \tag{16}$$

where

$$\lambda_0(\tau) \equiv \Lambda\tau\{1\} = 1-\frac{1}{2}E_2(\tau), \tag{17}$$

$$\lambda_j(\tau) \equiv \Lambda\tau\{E_j(\tau)\}\ (j \geqq 2), \tag{18}$$

and the following relation is used,

$$\Lambda\tau\{t\} = \tau+\frac{1}{2}E_3(\tau). \tag{19}$$

Comparing the expression (16) with the definition (1), we get the following approximate expansion for Hopf's function.

$$q(\tau) \fallingdotseq \frac{1}{2}E_3(\tau)+A_0\{1-\frac{1}{2}E_2(\tau)\}+\sum_{j=2}^{n} A_j\lambda_j(\tau), \tag{20}$$

This expansion is the approximate formula obtained from the lambda iterated variational method (hereafter referred to as the nth order lambda iterated formula), while the expansion (2) is the approximate formula obtained from the variational method (hereafter referred to as the nth order formula).

II.2. The Formulae Obtained by Kourganoff

Kourganoff (1952)[3] obtained the sixth order formula and the sixth order lambda iterated formula from the method described above. He chose as a sequence of m discrete τ values the following 16 values :

τ_k = 0.00 ; 0.01 ; 0.02 ; 0.03 ; 0.05 ; 0.10 ; 0.20 ; 0.30 ; 0.40 ; 0.60 ; 0.80 ; 1.00 ; 1.50 ; 2.00 ; 2.50 ; 3.00.

The values of $E_j(\tau)$'s, $\phi_j(\tau)$'s, and $\lambda_j(\tau)$'s were calculated numerically in the following way.

The $E_1(\tau)$ values were calculated by the following expansion,

$$E_1(\tau) = -\gamma - \log_e|\tau| + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{\tau^k}{k\cdot k!}, \tag{21}$$

where γ is the Euler-Mascheroni constant. The $E_j(\tau)$ values for $j \geqq 2$ were calculated by the following recurrence formula,

$$jE_{j+1}(\tau) = e^{-\tau} - \tau E_j(\tau) \ (j \geqq 1). \tag{22}$$

The values of $\phi_0(\tau)$, $\phi_1(\tau)$, and $\lambda_0(\tau)$ were calculated from the $E_j(\tau)$ values by the equations (8), (9), and (17), respectively. The values of $\phi_j(\tau)$'s and $\lambda_j(\tau)$'s for $j \geqq 2$ were calculated by the following equations,

$$\lambda_j(\tau) = \frac{1}{2}\left[M_j(\tau) - 2N_j(\tau) - \sum_{k=1}^{j-1} \frac{E_{k+1}(\tau)}{j-k}\right], \tag{23}$$

and

$$\phi_j(\tau) = 2\left[-M_{j+1}(\tau) + 2N_{j+1}(\tau) + 2E_{j+1}(\tau) + \sum_{k=1}^{j-1} \frac{E_{j+2}(\tau)}{j-k}\right], \tag{24}$$

In the above equations, the values of $M_j(\tau)$'s and $N_j(\tau)$'s for $j \geqq 1$ were calculated by the following recurrence formulae.

$$jM_j(\tau) = -\tau M_{j-1}(\tau) + e^{-\tau}\log_e 2 + (-1)^j e^{\tau} E_1(2\tau) + \sum_{k=0}^{j-1} (-1)^k E_{j-k}(\tau), \tag{25}$$

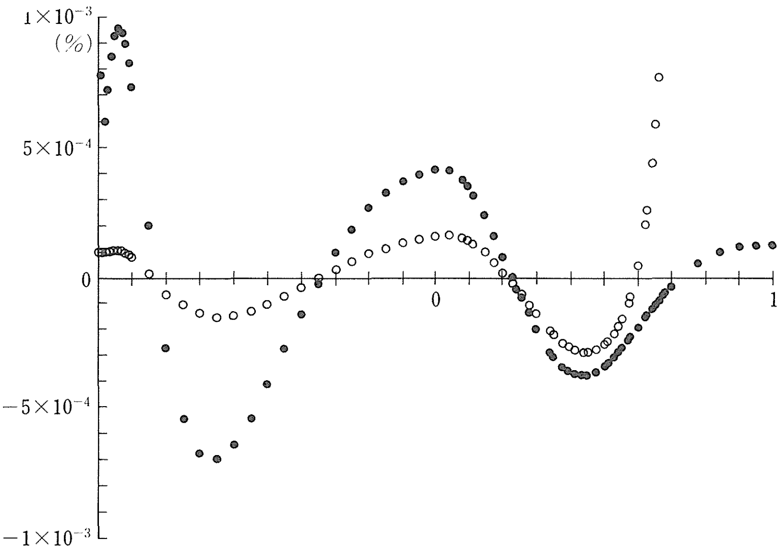

Fig. 1 The accuracy of the calculation by the sixth order formula and the sixth order lambda iterated formula obtained by Kourganoff (1952)[3]. The ordinate is the relative error and the abscissa is $\log_{10}\tau$. The open and filled circles represent the errors of the sixth order lambda iterated formula and of the sixth order formula, respectively.

and

$$jN_j(\tau) = -\tau N_{j-1}(\tau) - (\gamma + \log_e\tau)e^{-\tau} - \sum_{k=1}^{j} E_k(\tau) \tag{26}$$

The values of $M_0(\tau)$ and $N_0(\tau)$ were calculated by the following expansion,

$$M_0(\tau) = -(\gamma+\log_e\tau)E_1(-\tau) - \frac{1}{2}(\gamma+\log_e\tau)^2 + \frac{1}{6}\pi^2 + [E_1(\tau) - E_1(-\tau)]\log_e 2 - \sum_{k=1}^{\infty} a_k \frac{\tau^k}{k \cdot k!}, \tag{27}$$

and

$$N_0(\tau) = -(\gamma+\log_e\tau)E_1(\tau) - \frac{1}{2}(\gamma+\log_e\tau)^2 - \frac{1}{12}\pi^2 - \sum_{k=1}^{\infty}(-1)^k \frac{\tau^k}{k^2 \cdot k!},$$

Table 1 The $q(\tau)$ values with 9 significant figures which were calculated by Yoshioka (1985)[5] and are correct to within one unit in the 9th place of decimals

τ	$q(\tau)$	τ	$q(\tau)$	τ	$q(\tau)$
0.00	0.577350270	0.80	0.693533945	3.00	0.709807751
0.01	0.588235475	0.85	0.694986110	3.20	0.709955672
0.02	0.595390802	0.90	0.696293233	3.25	0.709986731
0.03	0.601241385	0.95	0.697472689	3.40	0.710068189
0.04	0.606286279	1.00	0.698539318	3.50	0.710114019
0.05	0.610757413	1.10	0.700383383	3.60	0.710154103
0.06	0.614788767	1.20	0.701908309	3.75	0.710205066
0.07	0.618467295	1.25	0.702571390	3.80	0.710219928
0.08	0.621853757	1.30	0.703177083	4.00	0.710270519
0.09	0.624992852	1.40	0.704238341	4.20	0.710309510
0.10	0.627918738	1.50	0.705130143	4.25	0.710317783
0.15	0.640133327	1.60	0.705882612	4.40	0.710339639
0.20	0.649550411	1.70	0.706519828	4.50	0.710352048
0.25	0.657119564	1.75	0.706801435	4.60	0.710362975
0.30	0.663366042	1.80	0.707061203	4.75	0.710376979
0.35	0.668616706	1.90	0.707522503	4.80	0.710381089
0.40	0.673091255	2.00	0.707916619	5.00	0.710395177
0.45	0.676945331	2.20	0.708543868	6.00	0.710430782
0.50	0.680293581	2.25	0.708673120	7.00	0.710441364
0.55	0.683223036	2.40	0.709007768	8.00	0.710444601
0.60	0.685801358	2.50	0.709193115	9.00	0.710445613
0.65	0.688082184	2.60	0.709353367	10.00	0.710445935
0.70	0.690108722	2.75	0.709554426		
0.75	0.691916258	2.80	0.709612455		

where a_k's are defined by the following expansion,

$$a_{2p}=2\left(1+\frac{1}{3}+\frac{1}{5}+\cdots\cdots+\frac{1}{2p-1}\right)+\frac{1}{2p}, \tag{28}$$

and

$$a_{2p+1}=2\left(1+\frac{1}{3}+\frac{1}{5}+\cdots\cdots+\frac{1}{2p+1}\right)+\frac{1}{2p+1}, \tag{29}$$

He obtained the following values for the coefficients A_0, A_2, ……, A_6 :

$$A_0=0.710447\,;\ A_2=-0.283903\,;\ A_3=0.642454\,;\ A_4=-1.224316$$
$$A_5=1.423034\,;\ A_6=-0.590226. \tag{30}$$

The accuracy of the calculation by the sixth order formula and the sixth order lambda iterated formula with these coefficients is shown in figure 1. This is obtained by comparing the calculated $q(\tau)$ values by these formulae with the $q(\tau)$ values calculated by Yoshioka (1985)[5]. The calculation by these formulae is done with a personal-computer PC-9801 VM 2 (NEC) with double precision floating number. The program is written in BASIC. The $q(\tau)$ vaslues calculated by Yoshioka (1985)[5] are listed in table 1. As is shown in figure 1, the accuracy of this sixth order lambda iterated formula is better than that of this sixth order formula for $\tau \leqq 4.0$ and vice versa for $\tau \geqq 4.2$. The errors in the $q(\tau)$ values calculated by the sixth order formula seem to be due to the inaccuracy of the formula, while the errors in the $q(\tau)$ values calculated by the sixth order lambda iterated formula seem to be due to the round off errors of the computer. The accuracy of this sixth order formula for $\tau \geqq 4.25$ is $\pm 1.5\times10^{-4}\%$ and that of this sixth order lambda iterated formula for $\tau \leqq 4.25$ is $\pm 2.6\times 10^{-4}\%$.

III. The Approximate Formula by Shinozuka

According to the method by Kourganoff (1952)[3], Shinozuka (1991)[6] obtained the eighth order formula and the eighth order lambda iterated formula. He chose as a sequence of m discrete τ values the same values as Kourganoff (1952)[3]. He calculated the values of $\phi_j(\tau)$'s and $\lambda_j(\tau)$'s also by the same formulae as Kourganoff (1952)[3]. He solved the system of normal equations by the iterative method using the initial value which was obtained by the direct method. The calculation was done with a personal computer PC-9801 with double precision floating number. The program was written in Quick BASIC.

He obtained the following values for the coefficients A_0, A_2, ……, A_8 :

$$A_0=0.7104460895988\,;\ A_2=-0.2851827012016\,;\ A_3=0.7187244221081$$
$$A_4=-2.1564496388019\,;\ A_5=5.4113602786473\,;\ A_6=-7.9073250106009$$
$$A_7=5.8963645405263\,;\ A_0=-1.6978115816004. \tag{31}$$

The accuracy of the calculation by the eighth order formula and the eighth order

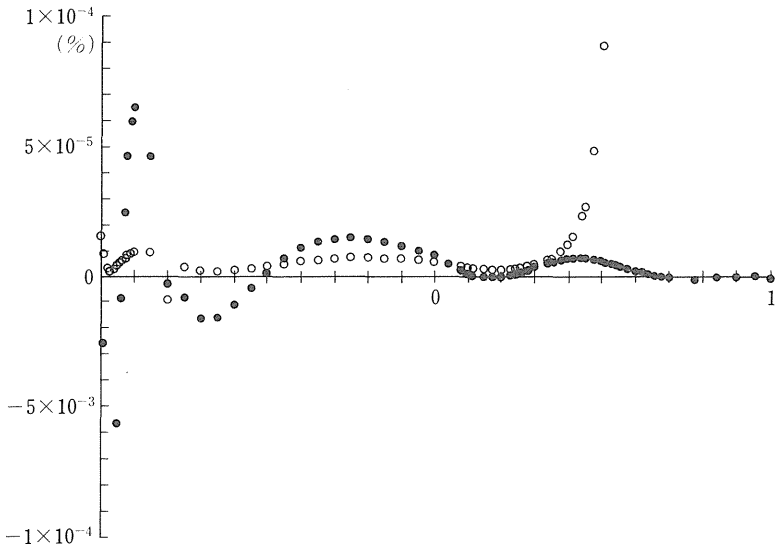

Fig. 2 Same as figure 1, but for the eighth order formula and the eighth order lambda iterated formula obtained by Shinozuka (1991)[6].

lambda iterated formula with these coefficients is shown in fiqure 2. This is obtained by the same process as figure 1. As is shown in figure 2, the accuracy of this eighth order lambda iterated formula is better than that of this eighth order formula for $\tau \leqq 1.1$ and vice versa for $\tau \geqq 1.2$. The accuracy of this eighth order formula for $\tau \geqq 1.1$ is $\pm 6.8 \times 10^{-6}$% and that of this eighth order lambda iterated formula for $\tau \leqq 2.6$ is $\pm 1.6 \times 10^{-5}$%. This accuracy is better by more than a factor of 10 than that of the formulae by Kourganoff (1952)[3].

IV. The Approximate Formula Modified in the Present Study

In this paper, the coefficients obtained by Shinozuka (1991)[6] have been modified. The modification is done according to the principle that the coefficients should minimize the following value ME,

$$ME = \sum_{r=1}^{s} \mid q_{cal}(\tau_r) - q_{ex}(\tau_r) \mid /s, \tag{32}$$

where $q_{ex}(\tau)$ is the exact $q(\tau)$ value and $q_{cal}(\tau)$ is the $q(\tau)$ value calculated according to the approximate formula (2) or (20). Hereafter, we represent the $q_{cal}(\tau)$ calculated according to the eighth order formula by $q_8(\tau)$ and the $q_{cal}(\tau)$ calculated according to the eighth order lambda iterated formula by $q_{i8}(\tau)$.

The $q(\tau)$ values listed in table 1 are chosen as the exact $q(\tau)$ values. For the eighth order formula, 29 values for $\tau > 2$ are chosen as a sequence of τ_r values, and for the eighth order lambda iterated formula, 41 values for $\tau \leqq 2$ are chosen. Thus, the coefficients obtained for the eighth order formula are different from those for the eighth order lambda iterated formula.

The coefficients which minimize the ME value are obtained in the following iterative way. First, the coefficients obtained by Shinozuka (1991)[6] are chosen as initial values, where his values are rounded off to 9 significant figures. Secondly, the A_0 value minimizing the ME value is obtained by calculation of the ME value with various A_0 values, where the other A_j values are kept the same values. Thirdly, the A_2 value minimizing the ME value is obtained in the same way as for the A_0 value, where the new A_0 value is taken as the value of A_0. Then, the A_3 value and so on minimizing the ME value are obtained in the same way. The above process is repeated until the A_j value agrees with that obtained by the previous process. The calculation is done also with a personal computer PC-9801 VM 2 with double precision floating number.

The following values are obtained for the eighth order lambda iterated formula :

$$A_0=0.710446072 ;\ A_2=-0.285182787 ;\ A_3=0.718724424 ;$$
$$A_4=-2.15644963 ;\ A_5=5.41136028 ;\ A_6=-7.90732501 ;$$
$$A_7=5.89636454 ;\ A_8=-1.69781157. \qquad (33)$$

On the other hand, the following values are obtained for the eighth order formula :

$$A_0=0.71044609 ;\ A_2=-0.285176651 ;\ A_3=0.71872176 ;$$
$$A_4=-2.1564497 ;\ A_5=5.41136028 ;\ A_6=-7.90733666 ;$$

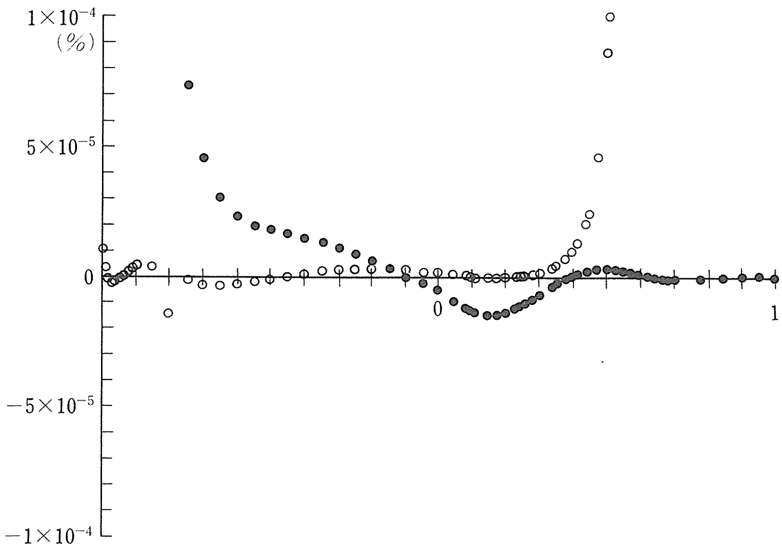

Fig. 3 Same as figure 1, but for the eighth order formula and the eighth order lambda iterated formula obtained in this paper.

$A_7 = 5.89636447$; $A_8 = -1.69781158$. (34)

The accuracy of the calculation by the eighth order formula and the eighth order lambda iterated formula with these coefficients is shown in figure 3. This is obtained by the same process as figure 1. As is shown in figure 3, the accuracy of this eighth order lambda iterated formula is better than that of this eighth order formula for $\tau \leqq 2.2$ and vice versa for $\tau \geqq 2.25$. The accuracy of this eighth order formula for $\tau \geqq 2.25$ is $\pm 2.9 \times 10^{-6}\%$ and that of this eighth order lambda iterated formula for $0.01 \leqq \tau \leqq 2.25$ is $\pm 4.1 \times 10^{-6}\%$. This accuracy is better by more than a factor of 3 than that of the formulae by Shinozuka (1991)[6].

V. Results and Discussion

In this paper, the coefficients obtained by Shinozuka (1991)[6] have been modified. With the modified coefficients, the $q_{i8}(\tau)$ values give the $q(\tau)$ values with the accuracy better than $\pm 4.1 \times 10^{-6}\%$ for $0.01 \leqq \tau \leqq 2.25$, and the $q_8(\tau)$ values give $q(\tau)$ values with the accuracy better than $\pm 2.9 \times 10^{-6}\%$ for $\tau \geqq 2.25$. This accuracy is better by more than 3 and 60 than that of the formulae by Shinozuka (1991)[6] and by Kourganoff (1952)[3], respectively. The use of these formulae do not increase the time of calculation by more than 25% as compared with the formulae by Kourganoff (1952)[3].

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