On difference Riccati equations and second order linear difference equations

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Abstract. In this paper, we treat difference Riccati equations and second order linear difference equations in the complex plane. We give surveys of basic properties of these equations which are analogues in the differential case. We are concerned with the growth and value distributions of transcendental meromorphic solutions of these equations. Some examples are given.

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1 introduction

The Nevanlinna theory has contributed to give a lot of applications to ordinary differential equations in the complex plane, see e.g., [9], [12]. Essence of research topics of nonlinear complex differential equations are contained in differential Riccati equations, and those of linear complex differential equations are contained in second order linear differential equations. Recently, the difference counterpart of the lemma on the logarithmic derivatives was obtained [3], [5], and then difference analogues of the Nevanlinna theory has developed, see e.g., [6], [13]. In this paper, we are concerned with the difference Riccati equation

$$\Delta f(z) + \frac{f(z)^2 + A(z)}{f(z) - 1} = 0, \tag{1.1}$$

and the linear difference equation of second order

$$\Delta^2 y(z) + A(z)y(z) = 0, \tag{1.2}$$

where $A(z)$ is a meromorphic function. For a function $\varphi(z)$, the difference operator Δ is defined by $\Delta\varphi(z) = \varphi(z+1) - \varphi(z)$. We define $\Delta^{n+1}\varphi(z) = \Delta(\Delta^n\varphi(z))$, $n = 1, 2, 3, \ldots$. We may assume that $f(z) \equiv 1$ is not a solution to (1.1). The equations (1.1) and (1.2) are respectively represented

$$f(z+1) = \frac{A(z) + f(z)}{1 - f(z)}$$

and

$$y(z+2) - 2y(z+1) + (A(z)+1)y(z) = 0.$$

This paper is constructed as follows. In Section 2, we collect summation formulas in the theory of difference equations and we give surveys of basic properties of (1.1) and (1.2). Section 3 is concerned with the growth and the value distribution of transcendental meromorphic solutions of (1.1) and (1.2).

2 Basic properties

Difference equatons have been studied in many aspects see e.g., [4], [10], [11]. Some expositions consider difference equatons in real domains, or discrete domains. Here, we mainly pay attention to considering meromprphic solutions of (1.1) and (1.2) in the complex plane with a rational coefficient $A(z)$.

2.1 Summation formulas

We define a symbol S which denotes a summation of $f(z)$. Namely, $\Delta(\mathsf{S}f(z)) = f(z)$. Basic ideas of summation can be found in e.g., [10, Pages 20–29], [11, Pages 80–83]. Let $Q(z)$ be an arbitrary periodic function of period 1. We have $\mathsf{S}(\Delta f(z)) = f(z) + Q(z)$. We define $z(0) = 1$ and $z(n) = z(z-1)\ldots(z-n+1)/n!$, $n = 1, 2, \ldots$. Then for a polynomial $b(z) = \sum_{\ell=0}^{M} b_\ell z^\ell$, there exist $\tilde{b}_\ell$, $\ell = 0, 1, \ldots, n$ such that $b(z) = \sum_{\ell=0}^{M} \tilde{b}_\ell z(\ell)$. The summation of $b(z)$ is given by $\mathsf{S}b(z) = \sum_{\ell=0}^{M} \tilde{b}_\ell z(\ell+1) + Q(z)$. We begin with a first order linear homogeneous equation, see e.g., [10, Page 48], [11, Pages 115–116].

Lemma 2.1 *Let $R(z)$ be a rational function. We write $R(z)$ in the form*

$$R(z) = \rho\,\frac{\prod_{k=1}^{n}(z - \alpha_k)}{\prod_{j=1}^{m}(z - \beta_j)},$$

where $\rho \neq 0$, α_k, $k = 1, \ldots, n$ and $j = 1, \ldots, m$ are complex numbers. The first order linear homogeneous equation

$$y(z+1) = R(z)y(z)$$

can be solved as

$$y(z) = Q(z)\rho^z \ \frac{\prod_{k=1}^{n} \Gamma(z-\alpha_k)}{\prod_{j=1}^{m} \Gamma(z-\beta_j)}. \tag{2.1}$$

The lemma below states that for any rational function $R(z)$ the summation $\mathsf{S}R(z)$ can be represented by $\Gamma(z)$, $\Psi(z)$ and their derivatives, where $\Psi(z)$ is the logarithmic derivative of $\Gamma(z)$, i.e., $\Psi(z) = \Gamma'(z)/\Gamma(z)$. See e.g., [11, Page 83]. We note that $\Gamma(z)$ and $\Psi(z)$ satisfies the following difference equations respectively.

$$\Gamma(z+1) = z\Gamma(z) \quad \text{and} \quad \Psi(z+1) = \Psi(z) + \frac{1}{z}. \tag{2.2}$$

Lemma 2.2 *Let $R(z)$ be a rational function. We write $R(z)$ in the form*

$$R(z) = \sum_{\ell=0}^{M} b_\ell z^\ell + \sum_{j=1}^{N} \sum_{k=1}^{n_j} \frac{c_{j,k}}{(z-\beta_j)^k}.$$

Then the summation of R is represented

$$\mathsf{S}R(z) = \ Q(z) + \sum_{\ell=0}^{M} \tilde{b}_\ell z(\ell+1) + \sum_{j=1}^{N} \sum_{k=1}^{n_j} \frac{(-1)^{k-1} c_{j,k}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \Psi(z-\beta_j). \tag{2.3}$$

2.2 Passages between Riccati equations and linear equations

It is known that a differential Riccati equation

$$w'(z) + w(z)^2 + A(z) = 0, \tag{2.4}$$

and a linear differential equation of second order

$$u''(z) + A(z)u(z) = 0, \tag{2.5}$$

are closely related by the passage

$$w(z) = -\frac{u'(z)}{u(z)}.$$

See e.g., [7, Pages 103–106].

We consider a passage between (1.1) and (1.2). For a nontrivial meromorphic solution $y(z)$ of (1.2), we set

$$f(z) = -\frac{\Delta y(z)}{y(z)}. \tag{2.6}$$

Then $f(z)$ satisfies the difference Riccati equation (1.1). In fact, from (2.6),

$$\begin{aligned} \Delta^2 y(z) &= -(\Delta f(z))y(z) - f(z+1)\Delta y(z) \\ &= -(\Delta f(z))y(z) + f(z+1)f(z)y(z). \end{aligned} \tag{2.7}$$

Combining (2.7) and (1.2), we have

$$\begin{aligned} &-(\Delta f(z))y(z) + f(z+1)f(z)y(z) + A(z)y(z) \\ &\qquad = -(f(z+1) - f(z))y(z) + f(z+1)f(z)y(z) + A(z)y(z) = 0, \end{aligned}$$

i.e.,

$$(-1 + f(z))f(z+1) + f(z) + A(z) = 0,$$

which implies (1.1). See e.g., [4, Pages 100-101].

Conversely, if (1.1) possesses a meromorphic solution $f(z)$, then a meromorphic solution $y(z)$ of first order difference equation (2.6) satisfies (1.2). In fact, from (2.7) and (1.1), we have

$$\begin{aligned} \Delta^2 y(z) &= (-f(z+1) + f(z) + f(z)f(z+1))y(z) \\ &= \left(\frac{-A(z) - f(z)}{1 - f(z)} + \frac{f(z) - f(z)^2}{1 - f(z)} + \frac{A(z)f(z) + f(z)^2}{1 - f(z)} \right) y(z) \\ &= -\frac{A(z)(1 - f(z))}{1 - f(z)} y(z) = -A(z)y(z), \end{aligned}$$

which implies (1.2).

Example 2.1 Let $a \in \mathbb{C}$, and set $A(z)$ below in (1.1) and (1.2)

$$A(z) = -\frac{2}{(z+a)(z+a+1)}.$$

The functions

$$f_1(z) = \frac{1}{z+a} \quad \text{and} \quad f_2(z) = -\frac{2}{z+a}$$

satisfy the difference Riccati equation (1.1). By Lemma 2.1, we obtain the corresponding solutions $y_1(z)$ and $y_2(z)$ of the linear difference equation (1.2)

$$y_1(z) = Q_1(z)\frac{\Gamma(z+a-1)}{\Gamma(z+a)} = Q_1(z)\frac{1}{z+a-1}$$

and

$$y_2(z) = Q_2(z)\frac{\Gamma(z+a+2)}{\Gamma(z+a)} = Q_2(z)(z+a)(z+a+1),$$

where $Q_1(z)$ and $Q_2(z)$ are periodic functions of order 1.

It is possible to construct a meromorphic solution of (1.1) other than $f_1(z)$ and $f_2(z)$. Set $y_3(z) = y_1(z) + y_2(z)$. Then $y_3(z)$ is a meromorphic solution of (1.2) and by (2.6) $f_3(z) = -\Delta y_3(z)/y_3(z)$ satisfies (1.1).

2.3 Difference Riccati equations

Let $\alpha_1(z)$, $\alpha_2(z)$ and $\alpha_3(z)$ be distinct meromorphic solutions of the differential Riccati equation (2.4). Then (2.4) possesses a one parameter family of meromorphic solutions $(f_c)_{c\in\mathbb{C}}$, see e.g., [1, Pages 371–373]. The following proposition is an analogue of this property.

Proposition 2.1 *Suppose that* (1.1) *possesses three distinct meromorphic solutions $f_1(z)$, $f_2(z)$ and $f_3(z)$. Then any meromorphic solution $f(z)$ of* (1.1) *can be represented by*

$$f(z) = \frac{f_1(z)f_2(z) - f_2(z)f_3(z) - f_1(z)f_2(z)Q(z) + f_1(z)f_3(z)Q(z)}{f_1(z) - f_3(z) - f_2(z)Q(z) + f_3(z)Q(z)}, \tag{2.8}$$

where $Q(z)$ is a periodic function of period 1. Conversely, if for any periodic function $Q(z)$ of period 1 we define a function $f(z)$ by (2.8)*, then $f(z)$ is a meromorphic solution of* (1.1).

Proof Let $h_j(z)$, $j = 1,2,3,4$ be distinct functions. We denote by $\mathcal{R}(h_1,h_2,h_3,h_4;z)$ a cross ratio of $h_j(z)$, $j=1,2,3,4$

$$\mathcal{R}(h_1,h_2,h_3,h_4;z) = \frac{h_1(z)-h_3(z)}{h_1(z)-h_4(z)}\bigg/\frac{h_2(z)-h_3(z)}{h_2(z)-h_4(z)}.$$

First we show that $f(z)$ is a meromorphic solution of the difference Riccati equation (1.1) if and only if $R(z+1) = R(z)$, where $R(z) = \mathcal{R}(f_1,f_2,f_3,f;z)$.

For the sake of simplicity, we write $(A(z)+f(z))/(1-f(z)) = L(f(z))$. Suppose that $f(z)$ is a meromorphic solution of (1.1). Then

$$\begin{aligned}\mathcal{R}(z+1) &= \frac{L(f_1(z))-L(f_3(z))}{L(f_1(z))-L(f(z))}\Bigg/\frac{L(f_2(z))-L(f_3(z))}{L(f_2(z))-L(f(z))}\\ &= \frac{\frac{(A(z)+1)(f_1(z)-f_3(z))}{(1-f_1(z))(1-f_3(z))}}{\frac{(A(z)+1)(f_1(z)-f(z))}{(1-f_1(z))(1-f(z))}}\frac{\frac{(A(z)+1)(f_2(z)-f(z))}{(1-f_2(z))(1-f(z))}}{\frac{(A(z)+1)(f_2(z)-f_3(z))}{(1-f_2(z))(1-f_3(z))}}\\ &= \mathcal{R}(z).\end{aligned}$$

Conversely, we suppose that $\mathcal{R}(z+1) = \mathcal{R}(z)$ so that

$$\frac{\frac{(A(z)+1)(f_1(z)-f_3(z))}{(1-f_1(z))(1-f_3(z))}}{\frac{A(z)+f_1(z)}{1-f_1(z)}-f(z+1)}\frac{\frac{A(z)+f_2(z)}{1-f_2(z)}-f(z+1)}{\frac{(A(z)+1)(f_2(z)-f_3(z))}{(1-f_2(z))(1-f_3(z))}} = \frac{f_1(z)-f_3(z)}{f_1(z)-f(z)}\frac{f_2(z)-f(z)}{f_2(z)-f_3(z)}.$$

Hence we obtain $f(z+1) = (A(z)+f(z))/(1-f(z))$, which concludes that $f(z)$ satisfies (1.1).

For any periodic function $Q(z)$ of period 1, we define $f(z)$ by

$$\mathcal{R}(f_1, f_2, f_3, f, z) = Q(z).$$

Then $f(z)$ is represented by (2.8), and $f(z)$ satisfies the difference Riccati equation (1.1). $\square$

This property is an analogue that a cross ratio of four distinct meromorphic solutions of a differential Riccati equation is a constant. See, e.g., [7, Pages 108–109].

Let $\alpha_1(z)$ and $\alpha_2(z)$ be distinct rational solutions of the differential Riccati equation (2.4). Two possibilities could be considered. One is that there is no other meromorphic solution, see e.g., [1, Page 396], and another is that there exist meromorphic solutions other than $\alpha_j(z)$, $j = 1, 2$. If there exists a rational solution $\alpha_3(z)$ distinct from $\alpha_j(z)$, $j = 1, 2$, then all meromorphic solutions of (2.4) are rational solutions. If there exists a transcendental meromorphic solution $w(z)$, then there is no rational solution other than $\alpha_j(z)$, $j = 1, 2$, see e.g., [1, Pages 393–394]. For difference Riccati equations, we have the following

Proposition 2.2 *Suppose that* (1.1) *possesses two distinct rational solutions* $a_1(z)$ *and* $a_2(z)$*. Then there exists a meromorphic solution* $a_3(z)$ *distinct from* $a_1(z)$ *and* $a_2(z)$ *so that any meromorphic solution* $f(z)$ *of* (1.1) *is represented in the form* (2.8).

Proof We set in (1.1)

$$f(z) = \frac{a_1(z)g(z) - a_2(z)}{g(z) - 1}. \tag{2.9}$$

Then we have

$$g(z+1) = \frac{a_1(z) - 1}{a_2(z) - 1} g(z).$$

By means of Lemma 2.1, we see that $g(z)$ is a meromorphic function as in the form (2.1) with a periodic function $Q(z)$ of period 1. From (2.9), we obtain a meromorphic solution $a_3(z)$ of (1.1). We choose a suitable $Q(z)$ so that $a_3(z)$ is distinct from $a_1(z)$ and $a_2(z)$. By Proposition 2.1, we conclude that any meromorphic solution of (1.1) is represented in the form (2.8). □

2.4 Linear difference equations of second order

Let $y_1(z)$ and $y_2(z)$ be meromorphic solutions of (1.2), and let $Q_1(z)$ and $Q_2(z)$ be periodic functions of period 1. Then the linear combination $Q_1(z)y_1(z) + Q_2(z)y_2(z)$ satisfies (1.2), which implies that the meromorphic solutions of (1.2) forms a vector space over the field of periodic functions of period 1. If there exist $Q_1(z)$ and $Q_2(z)$ periodic functions of period 1 such that $Q_1(z)y_1(z) + Q_2(z)y_2(z) = 0$, then we call $y_1(z)$ and $y_2(z)$ linearly dependent. Otherwise we call $y_1(z)$ and $y_2(z)$ linearly independent.

For functions $f(z)$ and $g(z)$, we denote by $\mathfrak{C}(z) = \mathfrak{C}(f, g; z)$ the Casoratian of $f(z)$ and $g(z)$, i.e.,

$$\mathfrak{C}(z) = \mathfrak{C}(f, g; z) = \begin{vmatrix} f(z) & g(z) \\ \Delta f(z) & \Delta g(z) \end{vmatrix} = \begin{vmatrix} f(z) & g(z) \\ f(z+1) & g(z+1) \end{vmatrix}. \tag{2.10}$$

It is known that $f(z)$ and $g(z)$ are linearly independent if and only if $\mathfrak{C}(f, g; z) \not\equiv 0$. See e.g., [11, Page 73].

Proposition 2.3 *If $y_1(z)$ and $y_2(z)$ are meromorphic solutions of (1.2), then the Casoratian $\mathfrak{C}(y_1, y_2; z)$ satisfies a difference equation*

$$\Delta \mathfrak{C}(z) = A(z)\mathfrak{C}(z). \tag{2.11}$$

Conversely, we assume that $y_1(z)(\not\equiv 0)$ and $y_2(z)$ satisfy (2.11). If $y_1(z)$ is a meromorphic solution of (1.2), then $y_2(z)$ is a meromorphic solution of (1.2).

Proof First we assert that for any functions $f(z)$ and $g(z)$, we have

$$\Delta\mathfrak{C}(f,g;z) = \begin{vmatrix} f(z+1) & g(z+1) \\ \Delta^2 f(z) & \Delta^2 g(z) \end{vmatrix}. \tag{2.12}$$

In fact,

$$\begin{aligned}
\Delta\mathfrak{C}(f,g;z) &= \begin{vmatrix} f(z+1) & g(z+1) \\ f(z+2) & g(z+2) \end{vmatrix} - \begin{vmatrix} f(z) & g(z) \\ f(z+1) & g(z+1) \end{vmatrix} \\
&= \begin{vmatrix} f(z+1) & g(z+1) \\ f(z+2)-2f(z+1) & g(z+2)-2g(z+1) \end{vmatrix} + \begin{vmatrix} f(z+1) & g(z+1) \\ f(z) & g(z) \end{vmatrix} \\
&= \begin{vmatrix} f(z+1) & g(z+1) \\ \Delta^2 f(z) & \Delta^2 g(z) \end{vmatrix}.
\end{aligned}$$

If $y_1(z)$ and $y_2(z)$ are meromorphic solutions of (1.2), then by (2.12)

$$\begin{aligned}
\Delta\mathfrak{C}(y_1,y_2;z) &= \begin{vmatrix} y_1(z+1) & y_2(z+1) \\ \Delta^2 y_1(z) & \Delta^2 y_2(z) \end{vmatrix} = \begin{vmatrix} y_1(z+1) & y_2(z+1) \\ -A(z)y_1(z) & -A(z)y_2(z) \end{vmatrix} \\
&= -A(z)\begin{vmatrix} y_1(z+1) & y_2(z+1) \\ y_1(z) & y_2(z) \end{vmatrix} = A(z)\mathfrak{C}(y_1,y_2;z).
\end{aligned}$$

The first assertion follows.

We assume that $y_1(z)$ and y_2 satisfy (2.11), i.e.,

$$\begin{vmatrix} y_1(z+1) & y_2(z+1) \\ \Delta^2 y_1(z) & \Delta^2 y_2(z) \end{vmatrix} = A(z)\begin{vmatrix} y_1(z) & y_2(z) \\ y_1(z+1) & y_2(z+1) \end{vmatrix}.$$

Then, we have

$$y_1(z+1)(\Delta^2 y_2(z) + A(z)y_2(z)) = y_2(z+1)(\Delta^2 y_1(z) + A(z)y_1(z)),$$

which gives the second assertion. □

The first assertion holds for n-th order linear homogeneous difference equations in general. See, e.g., [11, Page 79]. This property is a counter part that the Wronskian of linear independent meromorphic solutions of linear homogeneous differential equation satisfies a linear homogeneous differential equation of first order. See, e.g., [12, Pages 16–17].

Let $u_1(z)$ and $u_2(z)$ be linearly independent meromorphic solutions of (2.5). Denoting $c = W(u_1,u_2)$, we have $u_2(z) = h(z)u_1(z)$ with $h'(z) = c/u_1(z)^2$, see e.g., [2]. The following proposition is an analogue of this property.

Proposition 2.4 (i) *Let $y_1(z)$ and $y_2(z)$ be linearly independent meromorphic solutions of* (1.2)*, and let $\mathfrak{C}(z)$ be the Casoratian of $y_1(z)$ and $y_2(z)$. Then $y_2(z)$ is represented as $y_2(z) = g(z)y_1(z)$, in which $g(z)$ satisfies*

$$\Delta g(z) = \frac{\mathfrak{C}(z)}{y_1(z+1)y_1(z)}. \tag{2.13}$$

(ii) *Let $y_1(z)$ be a nontrivial meromorphic solution of* (1.2)*, and let $\mathfrak{C}(z)$ be a meromorphic solution of* (2.11)*. If $g(z)$ satisfies* (2.13)*, then $y_2(z) = g(z)y_1(z)$ is a meromorphic solution of* (1.2)*.*

Proof (i) From (2.10), $y_2(z)$ satisfies the following difference equation of first order

$$y_2(z+1) - \frac{y_1(z+1)}{y_1(z)}y_2(z) = \frac{\mathfrak{C}(z)}{y_1(z)}.$$

Write $y_2(z) = g(z)y_1(z)$. Then

$$g(z+1)y_1(z+1) - \frac{y_1(z+1)}{y_1(z)}g(z)y_1(z) = \frac{\mathfrak{C}(z)}{y_1(z)},$$

which implies (2.13).

(ii) Define $g(z) = y_2(z)/y_1(z)$. Then we have

$$\Delta^2 y_2(z) = g(z+2)y_1(z+2) - 2g(z+1)y_1(z+1) + g(z)y_1(z). \tag{2.14}$$

Using (2.13) and (2.11), we have

$$g(z+1) = g(z) + \frac{\mathfrak{C}(z)}{y_1(z+1)y_1(z)} \tag{2.15}$$

and

$$g(z+2) = g(z+1) + \frac{(A(z)+1)\mathfrak{C}(z)}{y_1(z+2)y_1(z+1)} \tag{2.16}$$

From (1.2), $y_1(z+2) = 2y_1(z+1) - (A(z)+1)y_1(z)$. Thus we combine (2.14), (2.15) and (2.16),

$$\begin{aligned}
\Delta^2 y_2(z) &= g(z+1)(y_1(z+2) - 2y_1(z+1)) + \frac{(A(z)+1)\mathfrak{C}(z)}{y_1(z+1)} + g(z)y_1(z) \\
&= \left(g(z) + \frac{\mathfrak{C}(z)}{y_1(z+1)y_1(z)}\right)(-(A(z)+1)y_1(z)) \\
&\quad + \frac{(A(z)+1)\mathfrak{C}(z)}{y_1(z+1)} + g(z)y_1(z) \\
&= -A(z)g(z)y_1(z) = -A(z)y_2(z),
\end{aligned}$$

which shows the assertion. $\square$

Suppose that $A(z)$ is a ratonal function in (1.1) and (1.2) and suppose that (2.11) admits a rational solution $\mathfrak{C}(z)$. Then Proposition 2.4 implies that if there exists a rational solution $y_1(z)$ of (1.2) then there exists a meromorphic solution $y_2(z)$ which is independent of $y_1(z)$. Hence under these conditions (1.1) has two distinct meromorphic solutions corresponding to $y_1(z)$ and $y_2(z)$.

Example 2.5 In (1.1) and (1.2), we set

$$A(z) = -\frac{2(55z^2+635z+1842)}{(z+2)(z+3)(z+4)(z+5)}.$$

Then

$$f_1(z) = \frac{-11z+58}{(z+2)(z+4)}$$

is a meromorphic solution of (1.1). The corresponding solution $y_1(z)$ of (1.2) satisfies (2.6), i.e.,

$$y_1(z+1) = \frac{(z+6)(z+11)}{(z+2)(z+4)}y_1(z).$$

By Lemma 2.1, we have

$$y_1(z) = Q_1(z)\frac{\Gamma(z+6)\Gamma(z+11)}{\Gamma(z+2)\Gamma(z+4)} = Q_1(z)(z+4)(z+5)\prod_{k=2}^{10}(z+k),$$

where $Q_1(z)$ is a periodic function of period 1. Let $\mathfrak{C}(z)$ be a meromorphic solution of (2.11), i.e.,

$$\mathfrak{C}(z+1) = \frac{(z-9)(z+6)^2(z+11)}{(z+2)(z+3)(z+4)(z+5)}\mathfrak{C}(z).$$

By Lemma 2.1, we obtain

$$\begin{aligned}\mathfrak{C}(z) &= Q_2(z)\frac{\Gamma(z-9)\Gamma(z+6)^2\Gamma(z+11)}{\Gamma(z+2)\Gamma(z+3)\Gamma(z+4)\Gamma(z+5)} \\ &= Q_2(z)(z+4)(z+5)\frac{\prod_{k=3}^{10}(z+k)}{\prod_{k=-9}^{1}(z+k)}\end{aligned}$$

where $Q_2(z)$ is a periodic function of period 1. We will obtain a meromorphic solution $y_2(z) = y_1(z)g(z)$ with $Q_1(z) \equiv 1$ and $Q_2(z) \equiv 1$ by (2.13), i.e.,

$$\Delta g(z) = \frac{\mathfrak{C}(z)}{y_1(z+1)y_1(z)} = \frac{1}{(z+6)\prod_{k=-9}^{11}(z+k)} = \frac{\alpha_6}{(z+6)^2} + \sum_{k=-9}^{11} \frac{\beta_k}{z+k},$$

where α_6 and β_k, $k = -9, -8, \ldots, 10, 11$ are nonzero constants. By means of Lemma 2.2, we obtain

$$g(z) = Q(z) + \tilde{\alpha}_6 \Psi'(z+6) + \sum_{k=-9}^{11} \tilde{\beta}_k \Psi(z+k),$$

where $Q(z)$ is a periodic function of period 1, and $\tilde{\alpha}_6$ and $\tilde{\beta}_k$, $k = -9, -8, \ldots, 10, 11$ are nonzero constants. Using (2.2), we see that $g(z)$ is represented by

$$g(z) = Q(z) + R_1(z) + \alpha\Psi'(z) + \beta\Psi(z), \tag{2.17}$$

where $R_1(z)$ is a rational function and α, β are constants. Set $Q(z) \equiv 0$ in (2.17) and we put $y_2(z) = (R_1(z) + \alpha\Psi'(z) + \beta\Psi(z))y_1(z)$. Then $y_2(z)$ is linearly independent of $y_1(z)$ and a transcendental meromorphic function. The corresponding solution $f_2(z)$ of (1.1) to $y_2(z)$ defined by (2.6) is also a transcendental meromorphic function.

3 Growth and Value distribution

In this section, we use the notations of the Nevanlinna theory, see e.g., [9], [12]. Let $m(r, h)$, $N(r, h)$ and $T(r, h)$ denote the proximity function, the counting function and the characteristic function of a meromorphic function $h(z)$ respectively. The growth order $\sigma(h)$ of $h(z)$ is defined by $\sigma(h) = \limsup_{r\to\infty} \log T(r, h)/\log r$. It is known that when treating the growth of meromorphic solutions of complex differential equations, the basic task is to find out their maximal growth, while in the case of difference equations, suitable solutions may grow arbitrarily fast, hence the basic task here is to find the minimal growth. Hence, we pay attention to transcendental meromorphic solutions of (1.1) and (1.2) having small order of growth.

Theorem 3.1 *Suppose that $A(z)$ is a rational function in* (1.1) *and suppose that* (1.1) *possesses a rational solution $a(z)$. Then* (1.1) *has no transcendental meromorphic solutions of order less than* $1/2$.

Proof Assume that (1.1) has a transcendental meromorphic solution $f(z)$ of order less than $1/2$. We define $k(z)$ from

$$f(z) = a(z) + \frac{1}{\alpha(z)k(z)}, \quad \text{with} \quad \alpha(z) = \frac{a(z-1)-1}{A(z-1)+1}.$$

We note that $T(r,k) = T(r,f) + O(\log r)$ and $\sigma(k) = \sigma(f)$. From (1.1), we have a difference equation of $k(z)$

$$k(z+1) = B(z)k(z) + 1, \quad \text{with} \quad B(z) = \frac{(a(z)-1)(a(z-1)-1)}{A(z-1)+1}. \tag{3.1}$$

We assert that $k(z)$ has finitely many poles. We take r_0 so large that zeros and poles of the rational function $B(z)$ are in $|z| < r_0$. If we assume that $k(z)$ has infinitely many poles, there is a pole z_0 of $k(z)$ such that z_0 is not in $|z| < r_0$. Hence, depending on the placement of z_0 in $|z| > r_0$, we conclude from (3.1) that $k(z)$ has poles either in $z_0 + n$, $n = 0, 1, 2 \ldots$ or in $z_0 - n$, $n = 0, 1, 2 \ldots$. Hence for a constant $H_0 > 0$, we have $n(r,k) \geq H_0 r$. Thus we have for some constant H

$$Hr \leq N(r,k) \leq T(r,k),$$

which implies that $\sigma(f) = \sigma(k) \geq 1$, a contradiction. We choose a rational function $b(z)$ such that $\tilde{k}(z) = b(z)k(z)$ is transcendental entire. From (3.1), $\tilde{k}(z)$ satisfies a first order nonhomogeneous difference equation with polynomial coefficients. From this we obtain a second order homogeneous difference equation of the form

$$P_2(z)\Delta^2\tilde{k}(z) + P_1(z)\Delta\tilde{k}(z) + P_0(z)\tilde{k}(z) = 0, \tag{3.2}$$

where $P_j(z)$, $j = 0, 1, 2$ are polynomials. Write p_j, $j = 0, 1, 2$ degrees of $P_j(z)$, $j = 0, 1, 2$ respectively. By means of an analogue of the Wiman–Valiron theory for the difference operator [8], at least one of the $p_1 - p_2 + 1$, $(p_0 - p_2 + 2)/2$ or $p_0 - p_1 + 1$ is a positive rational number less than $1/2$, which is impossible. $\square$

Theorem 3.2 *Suppose that $A(z)$ is a rational function in* (1.2)*. Then* (1.2) *has no transcendental meromorphic solutions of order less than* $1/2$*. Further we assume that* (1.2) *possesses a rational solution. Then every transcendental meromorphic solution of* (1.2) *has order of at least* 1.

Proof Assume that (1.2) admits a transcendental meromorphic $y(z)$ of order less than $1/2$. We discuss the similar arguments in the proof of Theorem 3.1. Since $A(z)$ is a rational function, there is no zeros and poles in

$|z| > r_0$ for sufficiently large r_0. If we assume $y(z)$ admits infinitely many poles, then there is a pole z_0 of $y(z)$ which is not contained in $|z| \leq r_0$. Hence, depending on the placement of z_0 in $|z| > r_0$, we conclude from (1.2) that $y(z)$ has poles either in $z_0 + p_n$, $n = 0, 1, 2\dots$ or in $z_0 - p_n$, $n = 0, 1, 2\dots$, where $\{p_n\}$ is a sequence of positive integers with $1 \leq p_{n+1} - p_n \leq 2$. This implies that $N(r, y) \geq Kr$ for some constant $K > 0$, which yields a contradiction. We choose a rational function $c(z)$ such that $\tilde{y}(z) = c(z)y(z)$ is transcendental entire. From (1.2), $\tilde{y}(z)$ satisfies the second order difference equation with polynomial coefficients of the form (3.2). Similarly to the proof of Theorem 3.1, we see that the assumption yields a contradiction.

We now suppose that (1.2) possesses a rational solution $\eta(z)$. By Proposition 2.4, for any transcendental meromorphic solution $y(z)$ of (1.2), we can write $y(z) = \eta(z)g(z)$, where $g(z)$ is a transcendental meromorphic function satisfying

$$\Delta g(z) = \frac{\mathfrak{C}(z)}{\eta(z+1)\eta(z)},$$

where $\mathfrak{C}(z) = \mathfrak{C}(\eta, y; z)$. If $\mathfrak{C}(z)$ is a rational function, then by Proposition 2.4 $g(z)$ is a meromorphic function of the form (2.3) since $\mathfrak{C}(z)/\eta(z+1)\eta(z)$ is a rational function. We have that $T(r, \Gamma(z)) = \frac{1}{\pi} r \log r(1 + o(1))$ and hence $T(r, \Psi(z)) = r(1 + o(1))$, see e.g., [16]. Hence $T(r, y) = T(r, g) + O(\log r) \geq K_1 r$ for some positive constant K_1, the assertion holds in this case. If $\mathfrak{C}(z)$ is a transcendental meromorphic function, by Yanagihara's inequality [15, Page 311 (2.4)] we have for large r

$$\begin{aligned} T(r, \mathfrak{C}(z)) &= T(r, \Delta g) + O(\log r) \leq T(r, \overline{g}) + T(r, g) + O(\log r) \\ &\leq 2T(r+1, g) + T(r, g) + O(\log r) \leq 3T(2r, g) + O(\log r), \end{aligned}$$

where $\overline{g}(z) = g(z+1)$. Since $A(z)$ is a rational function, by Proposition 2.3 $\mathfrak{C}(z)$ is written in the form (2.1). Using the growth properties of the Γ function and periodic functions if need, we have $T(r, \mathfrak{C}(z)) \geq K_2 r$ holds for large r, where K_2 is a positive constant. Hence there exists a positive constant K_3 such that $T(r, y) \geq K_3 r$ for large r. This concludes that $\sigma(y)$ is at least 1. $\square$

We note that a linear difference equation with polynomial coefficients of order $n \geq 3$ may possess a transcendental entire solution of order of growth less than 1/2, see e.g., [8].

Finally, we discuss relations between transcendental meromorphic solutions and rational solutions of Riccati equations.

Let $\alpha(z)$ be a rational function. Suppose that (2.4) possesses a transcendental meromorphic solution $w(z)$. If $\alpha(z)$ is not a solution of (2.4), then $w(z)-\alpha(z)$ has infinitely many zeros, see [14]. Concerning the counterpart of the difference Riccati equation (1.1), the corresponding property holds. We suppose that (1.1) possesses a transcendental meromorphic solution $f(z)$. Then $f(z)-a(z)$ has infinitely many zeros unless a rational function $a(z)$ solves (1.1), see [13]. We consider the case $\alpha(z)$ is a meromorphic solution of (2.4). Set $w(z)=v(z)+\alpha(z)$ in (2.4). Then we have $v'(z)=-v(z)(v(z)+2\alpha(z))$. We see that zeros of $v(z)$ must be poles of $\alpha(z)$, which implies that $v(z)$ has only finitely many zeros. Hence, we conclude that $w(z)-\alpha(z)$ has finitely many zeros.

We assert that in the difference case the analogue of this property does not always hold. Suppose that (1.1) has a rational solution $a(z)$. There may exist transcendental meromorphic solution $f_1(z)$ such that $f_1(z)-a(z)$ admits infinitely many zeros, and there may exist transcendental meromorphic solution $f_2(z)$ such that $f_2(z)-a(z)$ has only finitely many zeros. Below we give an example in which both cases occur.

Example 3.1 We consider the following difference Riccati equation

$$\Delta f(z)+\frac{f(z)^2+A(z)}{f(z)-1}=0, \text{ with } A(z)=-\frac{z^4-z^2+2z+3}{(z^2+z-1)(z^2+3z+1)},$$

which possesses a rational solution

$$a(z)=\frac{z^2-z+1}{z^2+z-1}.$$

We set

$$f(z)=a(z)+\frac{1}{\alpha(z)}\frac{1}{k(z)} \quad \text{with} \quad \alpha(z)=\frac{a(z-1)-1}{A(z-1)+1}=-\frac{z^2+z-1}{2z^2}.$$

Then $k(z)$ satisfies a nonhomogeneous first order difference equation

$$k(z+1)=\frac{z-1}{z^2}k(z)+1 \tag{3.3}$$

with the associated homogeneous difference equation

$$k_0(z+1)=\frac{z-1}{z^2}k_0(z). \tag{3.4}$$

We write $1/\Gamma(z)=\gamma(z)$. It is known that $\gamma(z)$ is a transcendental entire function, see e.g., [12]. The function $k_0(z)=\gamma(z)/(z-1)$ solves (3.4). Since a rational function $\varphi(z) = z/(z-1)$ is a solution of (3.3), general solutions $k(z)$ of (3.3) can be written

$$k(z) = Q(z)k_0(z) + \varphi(z),$$

where $Q(z)$ is a periodic function of period 1. If we choose an entire periodic function in place of $Q(z)$, then $k(z)$ has at most one pole. In this case, $f(z) - a(z)$ has only finitely many zeros. On the other hand, taking meromorphic periodic function having infinitely many poles for $Q(z)$, we have that $f(z) - a(z)$ has infinitely many zeros.

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