TRANSCENDENTAL MEROMORPHIC SOLUTIONS OF SOME ALGEBRAIC DIFFERENTIAL EQUATIONS

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(July 11, 2007)

Abstract

In this paper we treat transcendental meromorphic solutions of some algebraic differential equations. We consider the number of distinct transcendental meromorphic solutions. Algebraic relations between meromorphic solutions and comparisons of the growth of transcendental meromorphic solutions are also discussed.

2000 *Mathematics subject classification* (*Amer. Math. Soc.*): Primary 34A20, Secondary 30D35 .
Keywords and phrases: Meromorphic function, Algebraic differential equation, Nevanlinna theory .

1. Introduction

The binomial differential equation

$$(y')^n = R(z, y),$$

where n is a positive integer and $R(z, y)$ is a rational function in z and y, has been studied under the assumption that it has a transcendental meromorphic solution y in the complex plane (for example, Yosida [18], Laine [10]). The result due to Steinmetz [14], Bank and Kaufman [1] states that by a suitable Möbius transformation $v = (\alpha y + \beta)/(\gamma y + \delta)$, $\alpha\delta - \beta\gamma \neq 0$, the binomial equation is classified into the following six simple differential equations:

$$\begin{aligned}
v' &= a_2(z)v^2 + a_1(z)v + a_0(z) && \text{(I)}\\
(v')^2 &= a(z)(v - b(z))^2(v - \tau_1)(v - \tau_2) && \text{(II)}\\
(v')^2 &= a(z)(v - \tau_1)(v - \tau_2)(v - \tau_3)(v - \tau_4) && \text{(III)}\\
(v')^3 &= a(z)(v - \tau_1)^2(v - \tau_2)^2(v - \tau_3)^2 && \text{(IV)}\\
(v')^4 &= a(z)(v - \tau_1)^2(v - \tau_2)^3(v - \tau_3)^3 && \text{(V)}\\
(v')^6 &= a(z)(v - \tau_1)^3(v - \tau_2)^4(v - \tau_3)^5 && \text{(VI)}
\end{aligned}$$

Grant-in-Aid for Scientific Research, Japan Society for the Promotion of Science, (C) (1) (No. 16540202) .

where $\tau_1, \ldots, \tau_4$ are distinct constants and $a_j(z)(\not\equiv 0)$, $j = 0, 1, 2$, $a(z)$, $b(z)$ are rational functions. The result cited above of Steinmetz (Theorem 2 in [14]) was generalized to the case when $R(z, y)$ is rational in y with meromorphic coefficients by v. Rieth [13] and He-Laine [9].

Throughout this paper 'meromorphic' means 'meromorphic in the complex plane' and we use the standard notation of the Nevanlinna theory of meromorphic functions (for example, [6], [11], [12]).

We consider the following three problems for the equations (II) especially when $b(z)$ is a constant, say (II*), and (III). The equations (II*) and (III) are treated in Section 2 and in Section 3 respectively.

The first problem is to classify the equations by the number of transcendental meromorphic solutions. The differential equations (I)–(VI) do not always admit transcendental meromorphic solutions. It depends on the coefficients of the equations. We investigate how many transcendental meromorphic solutions the differential equations have and under what conditions they have an infinite number of transcendental meromorphic solutions. We have some results for the Riccati equation (I) concerning numbers of meromorphic solutions, for example, [2], or [11] Chapter 9. Answers of this problem for (II*) are given in Corollary to Theorem 2.2 (a), (b) and those for (III) are given in Corollary to Theorem 3.1 (a), (b).

The second problem is to find algebraic relations between meromorphic solutions. For the case of Riccati equation (I), four distinct solutions f_1, f_2, f_3, f_4 of (I) satisfy $\mathcal{R}(f_1, f_2, f_3, f_4) = c$, for a constant c, where $\mathcal{R}$ is a cross ratio of four elements (for example, [8], §4.2). We shall give an answer for (II*) by showing Theorem 2.1, and give an answer for (III) by showing Theorem 3.1 (iii).

The third problem is to compare the growth of transcendental meromorphic solutions. There are many results on the growth of transcendental meromorphic solutions of these six differential equations (for example, [1], [14], [15]). The fact proved in [1] and in [15] is that for the transcendental meromorphic solutions f of (II*) or (III), the order of f is a positive integral multiple of $1/2$, which is dependent on the coefficients of the equation. For example (*cf.* [15], Satz 1) for any solution of $(f')^2 = A(z)(f^2 - 1)$ the order of f is equal to $1 + d/2$ when $d \geq -1$, where

$$A(z) = c_1 z^d + c_2 z^{d-1} + \cdots \quad \text{for } z \to \infty,\ c_1 \neq 0.$$

This says that for given (fixed) coefficients transcendental meromorphic solutions f and g of the equation have the same order of growth.

We shall give more detailed estimates of growth of transcendental meromorphic solutions of (II*) by showing Theorem 2.1, and those of solutions of (III) by showing Corollary to Theorem 3.1 (c).

2. Results for the equation (II)

This section devotes to answer the problems, which we posed in Section 1, for the equation (II) when $b(z)$ is a constant, say (II*). The equation (II*) is changed to

$$(f')^2 = A(z)(f^2 - 1), \tag{1}$$

where $A(z) = (b - \tau_1)(b - \tau_2)a(z)$, by the linear transformation

$$f = 2(\tau_2 - b)(v - \tau_1)/((\tau_2 - \tau_1)(v - b)) - 1.$$

The equation (1) is more appropriate than (II*) for us to investigate its solutions. We denote by $\mathfrak{S}(A)$ the set of transcendental meromorphic solutions of (1) for a given rational function A, and denote by $\#\mathfrak{S}(A)$ the number of functions in $\mathfrak{S}(A)$.

In this section we prove the following theorems and corollaries:

Theorem 2.1. *Suppose that the differential equation* (1) *possesses distinct transcendental meromorphic solutions f and g. Then there is a constant c such that*

$$f^2 + 2cfg + g^2 = 1 - c^2. \tag{2}$$

Conversely, if there are two nonconstant meromorphic functions f and g satisfying (2)*, then the following relation holds:*

$$(f')^2/(f^2 - 1) = (g')^2/(g^2 - 1), \tag{3}$$

so that if f is a solution of (1)*, so is g.*

Corollary to Theorem 2.1. *Suppose that the differential equation* (1) *possesses transcendental meromorphic solutions f and g. Then we have*

$$T(r, g) = T(r, f) + O(1). \tag{4}$$

Theorem 2.2. *Suppose that the differential equation* (1) *admits at least three transcendental meromorphic solutions. Then we have:*

(i) *There is a rational function $\alpha(z)$ such that $A(z) = \alpha(z)^2$.*

(ii) *We can write $\alpha(z)$ in* (i) *as a decomposition of partial fractions*

$$\alpha(z) = p(z) + \sum_{j=1}^{n} k_j (z - \tau_j)^{-1}, \tag{5}$$

where $p(z)$ is a polynomial not identically equal to 0, k_j $(j = 1, \cdots, n)$ are integers, and τ_j $(j = 1, \ldots, n)$ are distinct constants. Moreover, for any transcendental meromorphic solution f, there exists a constant $C \in \mathbb{C}$ such that

$$f(z) = \cosh\Big(\int_0^z p(z)dz + \sum_{j=1}^{n} \log(z - \tau_j)^{k_j} + C\Big). \tag{6}$$

Corollary to Theorem 2.2. *We have*

(a) *Suppose that the differential equation (1) admits at least three transcendental meromorphic solutions. Then $\#\mathfrak{S}(A) = \infty$.*

(b) *For a rational function A, there are three possibilities on the number of transcendental meromorphic solutions of (1): $\#\mathfrak{S}(A) = 0$, $\#\mathfrak{S}(A) = 2$ or $\#\mathfrak{S}(A) = \infty$.*

We note that any nonconstant meromorphic solution f of (1) satisfies the second order linear differential equation

$$f'' - (A'/2A)f' - Af = 0. \tag{7}$$

In fact, differentiating (1), we have $2f'f'' = A'(f^2 - 1) + 2Aff'$. Combining this and (1), we obtain (7) since $f' \neq 0$.

For the proofs of Theorems 2.1 and 2.2, we need some lemmas given below.

Lemma 2.3 ([4], Theorem 1)**.** *Let F and G be meromorphic functions. F and G satisfy $F^2 + G^2 = 1$ if and only if there is a meromorphic function $\beta(z)$ such that*

$$F = 2\beta/(1 + \beta^2) \quad \text{and} \quad G = (1 - \beta^2)/(1 + \beta^2).$$

Lemma 2.4. *Let f be a nonconstant meromorphic function and put*

$$R(z) = (f')^2/(f^2 - 1). \tag{8}$$

If $R(z)$ has poles, any pole of $R(z)$ is of order at most 2.

Proof of Lemma 2.4 Given a pole z_0 of $R(z)$, z_0 is either a pole of f, a zero of $f(z) - 1$ or a zero of $f(z) + 1$. If z_0 is a pole of f, then a standard pole order comparison of (8) implies that $R(z)$ has a double pole at z_0. By a similar reasoning, if $f(z) = \pm 1 + \sum_{j=k}^{\infty} \alpha_j (z - z_0)^j$ around z_0, then $R(z)$ is regular at z_0 when $k \geq 2$, while $R(z)$ has a simple pole at z_0 when $k = 1$.

Lemma 2.5. *Suppose that a meromorphic function α is written in a neighborhood of a_0 as*

$$\alpha(z) = k/(z - a_0) + h(z), \quad (k \neq 0), \tag{9}$$

where $h(z)$ is regular at a_0. Then, the differential equation

$$w'' - (\alpha'(z)/\alpha(z))w' - \alpha^2(z)w = 0 \tag{10}$$

has a single-valued meromorphic solution in a neighborhood of a_0 if and only if k is equal to an integer.

Proof of Lemma 2.5 From (9), it is easy to see that a_0 is a regular-singular point for (10) (see [7], Satz 3.2). The corresponding indicial equation at a_0 is

$$\rho(\rho - 1) + \rho - k^2 = \rho^2 - k^2 = 0$$

and its solutions are $\rho = k$ and $\rho = -k$. Therefore, it is easy to see that (10) has a nonconstant meromorphic solution in a neighborhood of a_0 if and only if k is equal to an integer. □

Proof of Theorem 2.1 Assume that f and g are transcendental meromorphic solutions to (1), namely

$$(f')^2 = A(f^2 - 1) \quad \text{and} \quad (g')^2 = A(g^2 - 1). \tag{11}$$

Further it follows from (7) that

$$f'' - (A'/2A)f' - Af = 0 \quad \text{and} \quad g'' - (A'/2A)g' - Ag = 0. \tag{12}$$

Add two equations in (12), and then multiply the obtained equality by $2(f' + g')/A$ to obtain

$$\frac{2(f' + g')(f'' + g'')}{A} - \frac{A'}{A^2}(f' + g')^2 = 2(f + g)(f' + g'),$$

from which we have $((f' + g')^2/A)' = ((f + g)^2)'$ so that

$$(f' + g')^2/A = (f + g)^2 + c', \tag{13}$$

where c' is a constant. From (11) and (13) we eliminate A, f' and g' to obtain (2), where $c = 1 + c'/2$. Next we suppose that two nonconstant meromorphic functions f and g satisfy (2). When $c^2 = 1$, we have $f = \pm g$ and so the relation (3) holds. We consider the case $c^2 \neq 1$. Write (2) as

$$(f + cg)^2 + (1 - c^2)g^2 = 1 - c^2. \tag{14}$$

Differentiating the both sides of (14), we have

$$(f' + cg')(f + cg) + (1 - c^2)g'g = 0. \tag{15}$$

Combining (14) and (15), we obtain

$$(g')^2/(1 - g^2) = (f' + cg')^2/(1 - c^2)g^2. \tag{16}$$

Similarly we obtain by symmetry

$$(f')^2/(1 - f^2) = (g' + cf')^2/(1 - c^2)f^2. \tag{17}$$

We can write (15) as $f(f' + cg') = -g(g' + cf')$, so that the right-hand sides of (16) and (17) are equal, which results in the assertion. □

Proof of Corollary to Theorem 2.1 If $c^2 = 1$, then $f = \pm g$ and we have $T(r, f) = T(r, g)$. Hence we only treat the case $c^2 \neq 1$. From (2), we have

$$(f/g)^2 + 2cf/g + 1 = (1 - c^2)/g^2,$$

from which we have by Nevanlinna's first fundamental theorem

$$2T(r, g) = 2T(r, f/g) + O(1).$$

Changing roles of f and g, and using Nevanlinna's first fundamental theorem we obtain the relation

$$2T(r, f) = 2T(r, g/f) + O(1) = 2T(r, f/g) + O(1).$$

Combining the two relations above, we obtain (4). □

Proof of Theorem 2.2 (i) By the hypothesis of this theorem and by Theorem 2.1, there are transcendental meromorphic functions f and g satisfying

$$f^2 + 2cfg + g^2 = 1 - c^2 \quad (c^2 \neq 1),$$

from which we have

$$f^2 + ((cf + g)/\sqrt{1 - c^2})^2 = 1.$$

By Lemma 2.3, there is a meromorphic function $\beta(z)$ such that

$$f = 2\beta/(1 + \beta^2).$$

We see that $\beta(z)$ is transcendental since so is f. Hence we see that

$$A(z) = \frac{(f')^2}{f^2 - 1} = -\left(\frac{2\beta'}{1+\beta^2}\right)^2 = \left(\frac{2i\beta'}{1+\beta^2}\right)^2.$$

That is to say, $A(z) = \alpha(z)^2$ where $\alpha(z) = (2i\beta')/(1+\beta^2)$. Since $A(z)$ is a rational function, $\alpha(z)$ must be a rational function. $\square$

Proof of Theorem 2.2 (ii) By Theorem 2.2 (i), we can write $A(z) = \alpha(z)^2$ for a rational function $\alpha(z)$. If $\alpha(z)$ has a pole, then the pole is simple by Lemma 2.4 and the residue at the pole must be an integer by Lemma 2.5. Hence we can write $\alpha(z)$ in the following form:

$$\alpha(z) = p(z) + \textstyle\sum_{j=1}^{n} k_j (z - \tau_j)^{-1},$$

where $p(z)$ is a polynomial, n is the number of poles of $\alpha(z)$, k_j $(j = 1, \cdots, n)$ are integers, and τ_j $(j = 1, \ldots, n)$ are distinct constants. Put here $\zeta(z) = \int_0^z p(t)dt$. Then the meromorphic functions

$$f_1(z) = e^{\zeta(z)} \prod_{j=1}^{n} (z - \tau_j)^{k_j} \quad \text{and} \quad f_2(z) = e^{-\zeta(z)} \prod_{j=1}^{n} (z - \tau_j)^{-k_j}, \tag{18}$$

which are linearly independent, satisfy the linear differential equation (10). Since any solution $f(z)$ of (1) solves (10), $f(z)$ is written by a linear combination of f_1 and f_2, say

$$f(z) = C_1 f_1(z) + C_2 f_2(z), \tag{19}$$

where C_1 and C_2 are constants. As $f_1'(z) = \alpha(z) f_1(z)$, $f_2'(z) = -\alpha(z) f_2(z)$ and $f_1 f_2 = 1$ from (18), by substituting (19) into (1), we obtain that $C_1 C_2 = 1/4$. Therefore we see that for some $C \in \mathbb{C}$, $f(z)$ is represented in the form

$$f(z) = \cosh\big(\zeta(z) + \sum_{j=1}^{n} \log(z - \tau_j)^{k_j} + C\big).$$

It is immediately concluded that if $p(z) \equiv 0$, then meromorphic solutions to (1) are rational functions, which is a contradiction. Hence $p(z) \not\equiv 0$, the assertion follows. $\square$

Proof of Corollary to Theorem 2.2 (a): It follows from Proof of Theorem 2.2 (ii) if $A = \alpha^2$, where α satisfies (5), then a meromorphic function of the form (6) is a solution of (1). This implies that $\mathfrak{S}(A)$ is an uncountable set when $p(z) \not\equiv 0$. This implies that if (1) possesses at least three distinct transcendental meromorphic solutions, then $\#\mathfrak{S}(A) = \infty$.

(b): It is clear that if f is a transcendental meromorphic solution of (1), then $-f$ is also a transcendental meromorphic solution of (1). By (a), $\#\mathfrak{S}(A) \geq 3$ implies $\#\mathfrak{S}(A) = \infty$. Therefore we have proved (b). $\square$

Remark 2.6 We mention a condition which implies $\mathfrak{S}(A)$ is an empty set: *If A has at least one pole of order not less than 3, then $\#\mathfrak{S}(A) = 0$.* This is a direct consequence of Lemma 2.4.

We showed (a) of Corollary to Theorem 2.2 by means of Theorem 2.2. We mention here that we get the same result by only using the algebraic relation (2) and the relation (4). In fact, by the hypothesis of this Corollary, there are two meromorphic functions f and g in $\mathfrak{S}(A)$ satisfying

$$f^2 + 2cfg + g^2 = 1 - c^2 \quad (c^2 \neq 1),$$

from which we have

$$f^2 + ((cf + g)/\sqrt{1 - c^2})^2 = 1.$$

This shows that $(cf + g)/\sqrt{1 - c^2} \in \mathfrak{S}(A)$ by (3) in Theorem 2.1 and (4) since $f \in \mathfrak{S}(A)$. Put

$$h = (cf + g)/\sqrt{1 - c^2} \quad \text{and} \quad F = \gamma f + \delta h,$$

where γ and δ are constants satisfying $\gamma^2 + \delta^2 = 1$. Then

$$f^2 + h^2 = 1 \quad \text{and} \quad ff' = -hh'. \tag{20}$$

Now, we are going to prove that $F \in \mathfrak{S}(A)$. In fact, by (20)

$$\begin{aligned} (F')^2 &= \gamma^2 (f')^2 + 2\gamma\delta f'h' + \delta^2 (h')^2 \\ &= \gamma^2 A(f^2 - 1) + \delta^2 A(h^2 - 1) - 2\gamma\delta f(f')^2/h \\ &= A(\gamma^2 f^2 + \delta^2 h^2 - 1) + 2\gamma\delta A fh \\ &= A((\gamma f + \delta h)^2 - 1) \end{aligned} \tag{21}$$

since $(f')^2/h^2 = (h')^2/f^2 = -A$ by (20) and $f, h \in \mathfrak{S}(A)$. It follows from (21) that $F = \gamma f + \delta h$ is a meromorphic solution of (1) and by (4), $\gamma f + \delta h \in \mathfrak{S}(A)$. This proves the assertion.

3. Results for the equation (III)

In this section we are concerned with the differential equation of the type (III) in Section 1. It will be seen below that solutions of the equation (III) are closely connected with the Weierstrass $\wp$-function. We choose and fix a $\wp$-function satisfying

$$(\wp')^2 = 4\wp^3 - \tilde{g}_2 \wp - \tilde{g}_3, \tag{22}$$

where $\tilde{g}_2$, $\tilde{g}_3$, are constants satisfying $27\tilde{g}_3^2 - \tilde{g}_2^3 \neq 0$. For the sake of brevity we put $G(x) = 4x^3 - \tilde{g}_2 x - \tilde{g}_3$, and we denote by e_1, e_2, e_3 the distinct roots of $G(x) = 0$. For any solution v of (III), we set

$$f(z) = \frac{\alpha}{v(z) - \tau_4} - \beta \quad \text{with} \quad \alpha = -\frac{(\tau_1 - \tau_4)(\tau_2 - \tau_4)(\tau_3 - \tau_4)}{4},$$
$$\beta = \frac{1}{12}\big(2\tau_4(\tau_1 + \tau_2 + \tau_3) - (\tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1) - 3\tau_4^3\big).$$

Then the equation of type (III) can be translated into the following form:

$$(f')^2 = A(z)(4f^3 - \tilde{g}_2 f - \tilde{g}_3) = A(z)G(f), \tag{23}$$

where $A(z) \not\equiv 0$ is a rational function. We denote by $\mathfrak{T}(A)$ the set of transcendental meromorphic solutions of (23) for a given rational function A, and denote by $\#\mathfrak{T}(A)$ the number of functions in $\mathfrak{T}(A)$.

The purpose of this section is to show the following theorem and corollary:

Theorem 3.1. *Suppose that the equation* (23) *admits two transcendental meromorphic solutions f and g such that $f \neq L(g)$ for some Möbius transformation L such that $L(z) \not\equiv z$. Then we have*

(i) *There exists a polynomial $a(z)$ such that $A(z) = a'(z)^2$.*

(ii) *Any $f(z) \in \mathfrak{T}(A)$ can be written by*

$$f(z) = \wp(a(z) + c), \quad c \in \mathbb{C}, \tag{24}$$

where $\wp$ is the Weierstrass $\wp$ function given in (22).

(iii) *Let $u(z)$ and $v(z)$ denote arbitrary distinct transcendental meromorphic solutions of* (23). *Then there exists a constant $d_0 \in \mathbb{C}$, such that $U = u - d_0$ and $V = v - d_0$ satisfy an algebraic relation*

$$U^2V^2 - G_2UV - G_1(U + V) - G_0 = 0, \tag{25}$$

where G_0, G_1 and G_2 are constants.

Conversely if transcendental meromorphic functions U and V satisfy (25), *then we have*

$$(U')^2/K(U) = (V')^2/K(V), \tag{26}$$

where $K(x)$ is a polynomial of degree 3, expressed as

$$K(x) = 4x^3 + ((G_0 + G_2^2)/G_1)x^2 + 2G_2x + G_1. \tag{27}$$

Corollary to Theorem 3.1. *We have*

(a) *If the equation* (23) *admits two transcendental meromorphic solutions f and g such that $f \neq L(g)$ for some Möbius transformation L which is not identity, then $\#\mathfrak{T}(A) = \infty$.*

(b) *For a rational function A, there are three possibilities on the number of $\mathfrak{T}(A)$: $\#\mathfrak{T}(A) = 0$, $\#\mathfrak{T}(A) = 4$ or $\#\mathfrak{T}(A) = \infty$.*

(c) *For any transcendental meromorphic solutions f and g of* (23)*, we have*

$$T(r, g) = T(r, f) + S(r), \tag{28}$$

where $S(r)$ is small with respect to $T(r, f)$ and $T(r, g)$.

We need the following results due to Bank and Kaufman [1, Lemma 5], and due to Valiron [16].

Lemma 3.2. *Let $H(w)$ be a polynomial having constant coefficients, and let $w(z)$ be a nonconstant elliptic function of elliptic order q, which is a solution of the differential equation $(w')^q = H(w)$. Then we have*

(a) *If c_0 and c_1 are complex numbers satisfying $c_1^q = H(c_0)$, then there exists a complex number ζ such that $w(\zeta) = c_0$ and $w'(\zeta) = c_1$.*

(b) *Any solution of the differential equation $(w')^q = H(w)$ which is meromorphic and nonconstant in a region of the plane must be of the form $w(z + C)$ where C is a constant.*

The lemma given below is also needed for Proof of Theorem 3.1.

Lemma 3.3. *Suppose that* (23) *has distinct transcendental meromorphic solutions f and g. If f and g have a common pole z_0, then $\varphi := f - g$ does not have a zero at z_0.*

Proof of Lemma 3.3 We write A in a neighborhood of z_0 as

$$A(z) = R_A(z - z_0)^\lambda + O(z - z_0)^{\lambda+1}, \quad R_A \neq 0, \tag{29}$$

where λ is an integer. Let μ_f and μ_g denote orders of poles of f and g at z_0 respectively. From (23), $-2(\mu_f + 1) = \lambda - 3\mu_f$, that is, $\mu_f = 2 + \lambda$. Similarly we have $\mu_g = 2 + \lambda$. For the sake of brevity we write $\mu_f = \mu_g = \mu$.

Write f and g in a neighborhood of z_0 as

$$f(z) = \frac{R_f}{(z - z_0)^\mu} + O(z - z_0)^{-(\mu-1)}, \quad R_f \neq 0, \tag{30}$$

$$g(z) = \frac{R_g}{(z - z_0)^\mu} + O(z - z_0)^{-(\mu-1)}, \quad R_g \neq 0. \tag{31}$$

Substituting these representations into (23) and comparing the coefficients of terms $(z-z_0)^{-2(\mu+1)}$, we obtain

$$R_f = R_g = \mu^2/4R_A. \tag{32}$$

It follows from (23) that

$$(\varphi'/\varphi)(f'+g') = A\big(4(f^2+fg+g^2)-\tilde{g}_2\big). \tag{33}$$

Assume that φ has a zero at z_0 of order $\sigma > 0$. We compare the coefficients of $(z-z_0)^{-(\mu+2)}$ in the Laurent expansions in both sides of (33). Using (32), we obtain

$$\sigma\left(-\frac{\mu^3}{4R_A}-\frac{\mu^3}{4R_A}\right) = R_A\left(4\big(\frac{\mu^4}{16R_A^2}+\frac{\mu^4}{16R_A^2}+\frac{\mu^4}{16R_A^2}\big)\right),$$

that is, $-\sigma = 3\mu/2$, which is absurd. We have thus proved Lemma 3.3. □

Furthermore we mention a remark below to state some basic properties of solutions of (23).

Remark 3.4 (A) Every solution f of (23) satisfies

$$f'' = \frac{A'(z)}{2A(z)}f' + \frac{A(z)}{2}(12f^2-\tilde{g}_2). \tag{34}$$

Moreover, if f and g are distinct solutions of (23), then we have

$$\varphi'' - \frac{A'(z)}{2A(z)}\varphi' - 6A(z)(f+g)\varphi = 0, \quad \text{or} \tag{35}$$

$$\frac{\varphi''}{\varphi} - \frac{A'(z)}{2A(z)}\frac{\varphi'}{\varphi} = 6A(z)(f+g), \tag{36}$$

where $\varphi := f-g$.

(B) Let f be a transcendental meromorphic solution of (23). We introduce here the following four Möbius transformations:

$$L_0(x) = x, \quad L_1(x) = \frac{e_1x+e_1^2-e_2^2-e_1e_2}{x-e_1},$$
$$L_2(x) = \frac{e_2x+e_2^2-e_3^2-e_2e_3}{x-e_2}, \quad L_3(x) = \frac{e_3x+e_3^2-e_1^2-e_3e_1}{x-e_3}.$$

We see that $L_j(f)$, $j=0,1,2,3$ are also solutions of (23), which is verified by direct computations. Moreover, we assert that for any other Möbius transformation $L(x) = (ax+b)/(cx+d)$, $\Delta := ad-bc \neq 0$, the equation (23) is not solved by $L(f)$. To show this, we assume that $L(f)$ satisfies (23), that is,

$$\Delta^2\frac{(f')^2}{(cf+d)^4} = A(z)\left(4\big(\frac{af+b}{cf+d}\big)^3 - \tilde{g}_2\big(\frac{af+b}{cf+d}\big) - \tilde{g}_3\right). \tag{37}$$

First we treat the case $c = 0$. In this case we may assume that $d = 1$ and $a \neq 0$. Using (23) and (37), we eliminate f' and obtain a polynomial in f which must vanish. Then we have that $a = 1$ and $b = 0$ since f is a transcendental function. This implies that L must be L_0 in this case.

Next we consider the case $c \neq 0$. We may assume that $c = 1$ in this case. Using the same argument above, we obtain a polynomial in f of degree 4 which must vanish. Since f is transcendental, all coefficients must vanish. From the coefficients of f^4, f^3 and f^2, we obtain the following relations

$$4a^3 - \tilde{g}_2 a - \tilde{g}_3 = 0, \tag{38}$$

$$12a^2 b - 4b^2 + 4a^3 d + 8abd - 4a^2 d^2 - b\tilde{g}_2 - 3ad\tilde{g}_2 - 4d\tilde{g}_3 = 0, \tag{39}$$

and

$$4ab^2 + 4a^2 bd - bd\tilde{g}_2 - ad^2\tilde{g}_2 - 2d^2\tilde{g}_3 = 0. \tag{40}$$

From (38) and (40), we eliminate $\tilde{g}_3$. Then noting that $ad - b \neq 0$, we have

$$4ab + d(8a^2 - \tilde{g}_2) = 0. \tag{41}$$

(i) When $a \neq 0$, substituting $b = -d(8a^2 - \tilde{g}_2)/(4a)$ (from (41)) and $\tilde{g}_3 = 4a^3 - \tilde{g}_2 a$ (from (38)) into (39), we obtain

$$(a + d)d(12a^2 - \tilde{g}_2)^2 = 0.$$

We note that $d(12a^2 - \tilde{g}_2) \neq 0$. In fact, if $d(12a^2 - \tilde{g}_2) = 0$, by (41) we obtain that $a = 0$ since $ad - b \neq 0$, which is a contradiction. We have

$$d = -a \tag{42}$$

and from (41) we have

$$b = (8a^2 - \tilde{g}_2)/4. \tag{43}$$

(ii) When $a = 0$, we have $\tilde{g}_3 = 0$ by (38) and $d\tilde{g}_2 = 0$ by (41). If $d \neq 0$, $\tilde{g}_2 = 0$. This implies that $27\tilde{g}_3^2 - \tilde{g}_2^3 = 0$, which is a contradiction. We have $d = 0$. Substituting $a = 0, d = 0$ into (39), we obtain the equality

$$b(4b + \tilde{g}_2) = 0.$$

As $b \neq 0$ in this case ($ad - b \neq 0$), we have $b = -\tilde{g}_2/4$.

(i) and (ii) imply that (42) and (43) hold in any case.

By (38), we see that a coincides with one of the roots of $G(x) = 0$, say e_1, e_2 or e_3. We note that $\tilde{g}_2 = -4(e_1e_2 + e_2e_3 + e_3e_1)$ and $\tilde{g}_3 = 4e_1e_2e_3$. In view of (42) and (43) if $a = e_1$ then $b = e_1^2 - e_2^2 - e_1e_2$ and $d = -e_1$. This implies that L coincides with L_1. Similarly we see that $L = L_2$ when $a = e_2$, and $L = L_3$ when $a = e_3$.

Proof of Theorem 3.1 (i) Let f and g be two transcendental meromorphic solutions of (23) satisfying the hypothesis of this theorem. First we will show that $A(z)$ in (23) has no poles. From (23),

$$A(z) = (f')^2/G(f) = (g')^2/G(g) \quad \text{and} \quad G(f)/G(g) = (f'/g')^2. \tag{44}$$

Suppose that A has a pole z_0. From (44), there are four possibilities:
(i.1) z_0 is a pole of f and a pole of g,
(i.2) z_0 is a pole of f and a zero of $G(g)$,
(i.3) z_0 is a pole of g and a zero of $G(f)$,
(i.4) z_0 is a zero of $G(f)$ and a zero of $G(g)$.

Here we give a remark. In the cases (i.2)–(i.4) we consider the zero of $G(f)$ and $G(g)$. Assume that z_0 is a zero of $G(f)$. It gives that f has one of the e_j ($j = 1, 2, 3$) point at z_0. Without loss of generality we may assume that it is an e_1 point. We set $f_1 = L_1(f)$, where L_1 is given in Remark 3.4 (B), that is,

$$f_1 = (e_1 f + e_1^2 - e_2^2 - e_1 e_2)/(f - e_1). \tag{45}$$

Then we see by a simple computation that f_1 also satisfies (23) and z_0 is a pole of f_1. Hence the cases (i.2)–(i.4) reduce to the case (i.1), by using a suitable Möbius transformation which can be defined similar way to (45). Thus we have only to consider the case (i.1). Denote by μ_A, μ_f and μ_g orders of pole z_0 for A, f and g respectively.

From (23), we have $2(\mu_f + 1) = 3\mu_f + \mu_A$, that is, $(1 \leq)\mu_f = 2 - \mu_A$. Hence $\mu_f = \mu_A = 1$, similarly $\mu_g = 1$. Here we consider the Laurent expansions of A, f and g in a neighborhood of z_0 as follows:

$$\begin{aligned} A(z) &= R_A/(z - z_0) + \alpha_A + O(z - z_0), \quad R_A \neq 0, \\ f(z) &= R_f/(z - z_0) + \alpha_f + O(z - z_0), \quad R_f \neq 0, \\ g(z) &= R_g/(z - z_0) + \alpha_g + O(z - z_0), \quad R_g \neq 0. \end{aligned}$$

From (32), $R_f = R_g = 1/4R_A$. Further, substituting these representations into (23) and comparing the coefficients of terms $(z - z_0)^{-3}$, we have

$$\alpha_f = \alpha_g = -R_f \alpha_A/3R_A = -\alpha_A/(12R_A^2) \tag{46}$$

By the assumption of this lemma $\varphi := f - g$ does not vanish identically and by (46) φ has a zero at z_0. However by Lemma 3.3 it is impossible that φ has a zero at z_0, a contradiction.

Secondly we will show that all zeros of A are of even order. Let z_1 be a zero of A. From (44), if z_1 is a zero of f', (respectively g') and if z_1 is not a zero of $G(f)$, (respectively $G(g)$), then the order of zero of A at z_1 is an even integer. Hence we shall consider four possibilities:

(i.5) z_1 is a pole of f and a pole of g,
(i.6) z_1 is a pole of f, a zero of g' and a zero of $G(g)$,
(i.7) z_1 is a pole of g, a zero of f' and a zero of $G(f)$,
(i.8) z_1 a zero of f', a zero of $G(f)$, a zero of g' and a zero of $G(g)$.

We have only to treat the case (i.5). In fact, for the cases (i.6)–(i.8), as in the cases (i.2)–(1.4) given above by using suitable Möbius transformations they reduce to the case (i.5). We denote by λ the order of zero of A at z_1, and denote by μ_f and μ_g the orders of pole of f and g at z_1 respectively.

Similarly to Proof of Lemma 3.3, we obtain $(1 \leq)\lambda = \mu_f - 2 = \mu_g - 2$, which implies $\mu_f \geq 3$. Simply we write $\mu_f = \mu_g = \mu$.

Consider the Laurent expansions of A, f and g in a neighborhood of z_1. Denote by R_A the coefficient of $(z-z_1)^{\mu-2}$ in the expansion of A, and denote by R_f, R_g the coefficients of $(z-z_1)^{-\mu}$ in the expansions of f, g respectively. From (23) similarly to (32), we have

$$R_f = R_g = \mu^2/4R_A. \tag{47}$$

We see that the coefficient of the term $(z - z_1)^{-2}$ in the right-hand side of (36) is $6R_A(R_f + R_g) = 3\mu^2$ by (47).

We divide the behavior of φ at $z = z_1$ into three cases, that is, φ has a pole at z_1, φ has a zero at z_1, or φ does not have a pole nor a zero at z_1.

We first assume that φ has a pole at z_1 of order ν. Note that by (47) ν is at most $\mu - 1$. In the left-hand side of (36), the coefficient of double pole z_1 is $\nu(\nu+1)+(\mu-2)\nu/2 = \nu^2+\mu\nu/2$. Hence we have $2\nu^2+\mu\nu-6\mu^2 = 0$, i.e., $\nu = -2\mu$ or $2\nu = 3\mu$. Since μ and ν are positive, $\nu = -2\mu$ is absurd. If $2\nu = 3\mu$, then we have that $\mu \leq -2$ using $\nu \leq \mu - 1$, which is also absurd.

Next we treat the case φ has a zero at z_1. By the assumption, $\varphi := f - g$ does not vanish. Hence in view of Lemma 3.3 this case does not occur.

Finally we consider the case φ does not have a pole nor a zero at z_1. In this case z_1 is a simple pole or a regular point of the left-hand side of (36). However the right-hand side has a double pole, a contradiction.

Therefore A must be a polynomial whose zeros are of even order, which implies that there exists a polynomial a such that $A = (a')^2$. □

Proof of Theorem 3.1 (ii) We follow the idea in the proofs of Lemma 3.2 (a) and (b), see Bank and Kaufman [1]. Let f be a transcendental meromorphic solution of (23). We fix $z_0 \in \mathbb{C}$ which is not a pole of f satisfying the conditions $a'(z_0) \neq 0$, $\wp'(z_0) \neq 0$ and $f'(z_0) \neq 0$ (or $G(f(z_0)) \neq 0$). Denote by D_0 a fundamental parallelogram of $\wp$ that contains z_0. Further we set $f(z_0) = b_0$ and $f'(z_0)/a'(z_0) = b_1$. Then from (23), $b_1^2 = G(b_0)$. In view of Lemma 3.2, there exists $z_1 \in D_0$ such that $\wp(z_1) = b_0$ and $\wp'(z_1) = b_1$. We set $\alpha(z) = a(z)+z_1-a(z_0)$ and $f_1 = f_1(z) = \wp(\alpha(z)) = \wp(a(z)+z_1-a(z_0))$. Then it holds $f_1'(z) = \wp'(\alpha(z))\alpha'(z) = \wp'(\alpha(z))a'(z)$, and hence

$$(f_1')^2 = (\wp'(\alpha))^2(a')^2 = AG(\wp(\alpha)) = AG(f_1)$$

which implies that f_1 is a meromorphic solution of (23). We have that

$$f_1(z_0) = \wp(a(z_0) + z_1 - a(z_0)) = \wp(z_1) = b_0 = f(z_0), \tag{48}$$

$$\begin{aligned} f_1'(z_0) &= \wp'(a(z_0) + z_1 - a(z_0))a'(z_0) \\ &= \wp'(z_1)a'(z_0) = b_1 a'(z_0) = f'(z_0). \end{aligned} \tag{49}$$

Set $\psi = f - f_1$. Then from (48) and (49) we have that $\psi(z_0) = \psi'(z_0) = 0$. We see that $A'/2A$ and A is analytic at z_0 from our assumption. Regarding g as to f_1 and φ as to ψ in (35), we conclude that $\psi = 0$, that is, f and f_1 must coincide. This proves (ii). □

Proof of Theorem 3.1 (iii) Let u and v denote meromorphic solutions of (23), and let $a(z)$ be a polynomial given in (i). We may assume that $u = u(z) = \wp(a(z))$ and we can write $v = v(z) = \wp(a(z) + c)$ for a constant $c \in \mathbb{C}$ by (ii). Put $\wp(c) = d_0$ and $\wp'(c) = d_1$. Then by the addition formula of $\wp$-function,

$$\wp(a(z) + c) = \frac{1}{4}\left(\frac{\wp'(a(z)) - \wp'(c)}{\wp(a(z)) - \wp(c)}\right)^2 - \wp(a(z)) - \wp(c),$$

that is,

$$v = \frac{1}{4}\left(\frac{\wp'(a(z)) - d_1}{u - d_0}\right)^2 - u - d_0. \tag{50}$$

Since $d_1^2 = G(d_0)$ and $(a'(z))^2 = A(z)$, from (23) and (50) we obtain

$$\Big(4(v + u + d_0)(u - d_0)^2 - G(u) - G(d_0)\Big)^2 = 4G(d_0)G(u). \tag{51}$$

Put $U = U(z) = u(z) - d_0$ and $V = V(z) = v(z) - d_0$. Then since $G(d_0) = 4d_0^3 - \tilde{g}_2 d_0 - \tilde{g}_3$ and $G'(d_0) = 12d_0^2 - \tilde{g}_2$, we can write (51) as

$$U^2V^2 - \frac{1}{2}G'(d_0)UV - G(d_0)(U + V) + \frac{1}{16}(G'(d_0)^2 - 48d_0G(d_0)) = 0,$$

which confirms that U and V satisfy a relation of the form (25).

Conversely we suppose that the relation (25) holds for meromorphic functions U and V. We differentiate (25) to obtain

$$U'(2UV^2 - G_2V - G_1) = -V'(2VU^2 - G_2U - G_1). \tag{52}$$

Using (25) we have

$$\begin{aligned} (2VU^2 - G_2U - G_1)^2 =& 4U^2(G_2UV + G_1(U + V) + G_0) + G_2^2U^2 \\ & + G_1^2 - 4U^3VG_2 - 4U^2VG_1 + 2G_2G_1U \\ =& 4G_1U^3 + (4G_0 + G_2^2)U^2 + 2G_1G_2U + G_1^2 \\ =& G_1K(U). \end{aligned} \tag{53}$$

Similarly we obtain

$$(2UV^2 - G_2V - G_1)^2 = G_1K(V). \tag{54}$$

Combining (52), (53) and (54), we obtain the assertion (26) with (27). □

Proof of Corollary to Theorem 3.1 (a): We can see (a) from (ii) of Theorem 3.1.

(b): Suppose that (23) has a transcendental meromorphic solution f. In the case there exists a transcendental meromorphic solution g of (23) such that $g \neq L(f)$ for some Möbius transformation, we have that $\#\mathfrak{T}(A) = \infty$ by (a). For the proof of (b) it remains to find the number of Möbius transformations L_j such that $L_j(f)$ satisfy the equation (23) if $\#\mathfrak{T}(A) \neq 0, \infty$. By means of Remark 3.4(B), the number of such Möbius transformations is equal to four, namely, $\#\mathfrak{T}(A) = 4$.

(c): In the case $f = L(g)$ for a Möbius transformation L, we have $T(r, f) = T(r, g) + O(1)$ by means of the Nevanlinna first fundamental theorem. We may suppose that $f \neq L(g)$ for any Möbius transformation L. Then in view of Theorem 3.1(iii), for a $d_0 \in \mathbb{C}$, $f_0 = f - d_0$ and $g_0 = g - d_0$ satisfy an algebraic relation (25). Since $T(r, f_0) = T(r, f) + O(1)$ and $T(r, g_0) = T(r, g) + O(1)$, it is enough to show that f_0 and g_0 satisfy the assertion of (c), namely $T(r, f_0) = T(r, g_0) + O(1)$. If $G_1 = 0$ in (25), then f_0g_0 is a constant, from which we obtain that $T(r, f_0) = T(r, g_0) + O(1)$. In what follows, we assume that $G_1 \neq 0$. Define meromorphic functions

$$f_1 = -(G_1g_0 + G_0)/f_0g_0^2 \quad \text{and} \quad g_1 = -(G_1f_0 + G_0)/g_0f_0^2. \tag{55}$$

From (24) for $U = f_0, V = g_0$, we have

$$f_0 - (G_2g_0 + G_1)/g_0^2 = (G_1g_0 + G_0)/f_0g_0^2.$$

Eliminating f_0 of this equation by using the first one of (54), we see that f_1 and g_0 satisfy (25). Similarly we see that f_0 and g_1 satisfy (25). Namely,

$$f_1^2g_0^2 - G_2f_1g_0 - G_1(f_1 + g_0) - G_0 = 0, \tag{56}$$

$$f_0^2g_1^2 - G_2f_0g_1 - G_1(f_0 + g_1) - G_0 = 0. \tag{57}$$

Thus f_0, g_0, f_1 and g_1 are transcendental meromorphic solutions of

$$(w')^2 = A(z)K(w), \tag{58}$$

where $A(z)$ is given in (23) and $K(w)$ is given in (27). We also have

$$f_0 + f_1 = (G_2g_0 + G_1)/g_0^2 \quad \text{and} \quad g_0 + g_1 = (G_2f_0 + G_1)/f_0^2. \tag{59}$$

It follows from (59) and $G_1 \neq 0$ that

$$2T(r, g_0) \leq T(r, f_0) + T(r, f_1) + O(1). \tag{60}$$

Using (55) and (59), we obtain

$$f_0^{-1} + f_1^{-1} = -(G_2 g_0 + G_1)/(G_1 g_0 + G_0). \tag{61}$$

By means of the first fundamental theorem of Nevanlinna and (61),

$$T(r, f_1) \leq T(r, g_0) + T(r, f_0) + O(1). \tag{62}$$

Combining (60) and (62), we have $T(r, g_0) \leq 2T(r, f_0) + O(1)$. Changing the roles of $f_0(z)$ and $g_0(z)$, we obtain $T(r, f_0) \leq 2T(r, g_0) + O(1)$. This implies that if $\varphi(r) = S(r, f_0)$, then $\varphi(r) = S(r, g_0)$, and if $\varphi(r) = S(r, g_0)$, then $\varphi(r) = S(r, f_0)$. Hence for two meromorphic functions f and g, we can write $S(r, f) = S(r)$ and $S(r, g) = S(r)$.

We recall that some properties of a transcendental meromorphic solution $w(z)$ of (58). Let $w(z)$ be a transcendental meromorphic solution of (58). Then by means of Gol'dberg's theorem [5], we see that $w(z)$ is of finite order. We have that all poles of $w(z)$ are double with a finite number of exceptions and $m(r, w) = O(\log r)$. All zeros of $w(z)$ are simple with a finite number of exceptions and $m(r, 1/w) = O(\log r)$ since we assume $G_1 \neq 0$. Hence,

$$N(r, w) = 2\overline{N}(r, w) + O(\log r) = T(r, w) + O(\log r), \tag{63}$$

and

$$N(r, 1/w) = \overline{N}(r, 1/w) + O(\log r) = T(r, w) + O(\log r). \tag{64}$$

Let z_0 be a pole of $f_0(z)$, and let z_1 be a pole of $f_1(z)$. Then we see from (25) and (56) (or (55)), z_0 is a zero of g_0, and z_1 is also a zero of g_0. If both $f_0(z)$ and $f_1(z)$ have a common double pole z_2, then z_2 is a zero of $g_0(z)$ of multiplicity at least two. From (64), the counting function of such common poles is of $O(\log r)$. Thus it concludes that

$$\overline{N}(r, f_0) + \overline{N}(r, f_1) \leq \overline{N}(r, 1/g_0) + O(\log r). \tag{65}$$

From (63), (64) and (65),

$$T(r, f_0) + T(r, f_1) \leq 2T(r, g_0) + O(\log r).$$

Combining this and (60), we obtain

$$T(r, f_0) + T(r, f_1) = 2T(r, g_0) + O(\log r). \tag{66}$$

Further we define

$$g_2 = -(G_1 f_1 + G_0)/g_0 f_1^2 \quad \text{and} \quad f_2 = -(G_1 g_1 + G_0)/f_0 g_1^2.$$

Repeating this process, we define sequences of meromorphic functions f_0, g_1, f_2, g_3, ..., and g_0, f_1, g_2, f_3, ... Namely we set for $k = 0, 1, 2, \ldots$,

$$f_{2k+3} = -\frac{G_1 g_{2k+2} + G_0}{f_{2k+1} g_{2k+2}^2}, \quad g_{2k+2} = -\frac{G_1 f_{2k+1} + G_0}{g_{2k} f_{2k+1}^2},$$

$$g_{2k+3} = -\frac{G_1 f_{2k+2} + G_0}{g_{2k+1} f_{2k+2}^2}, \quad f_{2k+2} = -\frac{G_1 g_{2k+1} + G_0}{f_{2k} g_{2k+1}^2}.$$

Then we see that all functions

$$\{f_j(z)\} \ \ (j = 0, 1, \dots) \quad \text{and} \quad \{g_k(z)\} \ \ (k = 0, 1, \dots)$$

are transcendental and satisfy the differential equation (58), all pairs

$$(f_j(z), g_{j+1}(z)) \quad \text{and} \quad (g_j(z), f_{j+1}(z)) \ \ (j = 0, 1, \dots)$$

satisfy (25) and that all triples

$$(f_{j-1}(z), f_{j+1}(z), g_j(z)) \quad \text{and} \quad (g_{j-1}(z), g_{j+1}(z), f_j(z)) \ \ (j = 1, 2, \dots)$$

satisfy (66). We write for $j = 0, 1, 2, \dots,$

$$h_j(z) = \begin{cases} f_j(z), & \text{if } j \text{ is odd} \\ g_j(z), & \text{if } j \text{ is even.} \end{cases}$$

Let a_0, b_0, a_1 and b_1 be positive constants. We assume that there exists a sequence $\{r_n\}$, $r_n \to \infty$ as $n \to \infty$ satisfying

$$\left\{ \begin{array}{ll} T(r_n, h_0) & \le a_0 T(r_n, f_0) + O(\log r_n) \\ T(r_n, h_0) & \ge b_0 T(r_n, f_0) + O(\log r_n) \end{array} \right. \tag{67}$$

and

$$\left\{ \begin{array}{ll} T(r_n, h_1) & \le a_1 T(r_n, f_0) + O(\log r) \\ T(r_n, h_1) & \ge b_1 T(r_n, f_0) + O(\log r). \end{array} \right. \tag{68}$$

We assert that there exist sequences $\{a_j\}$ and $\{b_j\}$, $j = 0, 1, 2, \dots,$ such that

$$T(r_n, h_j) \le a_j T(r_n, f_0) + O(\log r_n) \tag{69}$$

and

$$T(r_n, h_j) \ge b_j T(r_n, f_0) + O(\log r_n). \tag{70}$$

In view of (66) and the comment that we posed after the definitions of $\{f_j(z)\}$ and $\{g_j(z)\}$, we have for $j = 1, 2, \dots,$

$$T(r_n, h_{j-1}) + T(r_n, h_{j+1}) = 2T(r_n, h_j) + O(\log r_n). \tag{71}$$

Assume that (69) and (70) hold for $j = 0, 1, 2, \dots, k$. Then from (71),

$$\begin{aligned} T(r_n, h_{k+1}) &= 2T(r_n, h_k) - T(r_n, h_{k-1}) + O(\log r_n) \\ &\le 2a_k T(r_n, f_0) - b_{k-1} T(r_n, f_0) + O(\log r_n), \end{aligned}$$

which gives

$$a_{k+1} = 2a_k - b_{k-1}. \tag{72}$$

Similarly, we obtain

$$b_{k+1} = 2b_k - a_{k-1}. \tag{73}$$

Therefore, using the assumptions (67) and (68), we obtain $\{a_n\}$ which satisfies (69) and $\{b_n\}$ which satisfies (70) recursively by (72) and (73).

We now compute a_k and b_k concretely. Put $c_k = a_k + b_k$. Then we have that $c_{k+1} - 2c_k + c_{k-1} = 0$, and hence $c_k = (c_1 - c_0)k + c_0$, $k = 0, 1, 2, \ldots$. Thus we obtain

$$a_{k+1} - 2a_k - a_{k-1} = \mu k + \nu, \tag{74}$$

where $\mu = c_0 - c_1$ and $\nu = c_1 - 2c_0$. In (74), we set $d_k = a_{k+1} - a_k$. Then we have $d_k - 2d_{k-1} - d_{k-2} = \mu$. Further, we put $e_k = d_k + \mu/2$. Then

$$e_k - 2e_{k-1} - e_{k-2} = 0. \tag{75}$$

Thus we can write e_k with some constants γ_1 and γ_2:

$$e_k = \gamma_1 \lambda_1^k + \gamma_2 \lambda_2^k, \tag{76}$$

where $\lambda_1 = 1 + \sqrt{2}$ and $\lambda_2 = 1 - \sqrt{2}$, (roots of the equation $t^2 - 2t - 1 = 0$), see for example [3]. Thus $d_k = e_k - \mu/2$, and hence for $k = 1, 2, \ldots,$

$$\begin{aligned} a_k &= \sum_{j=0}^{k-1} d_j + a_0 = \sum_{j=0}^{k-1} \left(e_j - \frac{\mu}{2} \right) + a_0 \\ &= \sum_{j=0}^{k-1} \left(\gamma_1 \lambda_1^j + \gamma_2 \lambda_2^j - \frac{\mu}{2} \right) + a_0 \\ &= \gamma_1 \frac{1 - \lambda_1^k}{1 - \lambda_1} + \gamma_2 \frac{1 - \lambda_2^k}{1 - \lambda_2} - \frac{\mu}{2} k + a_0, \end{aligned} \tag{77}$$

and

$$\begin{aligned} b_k = 2a_k - a_{k+1} = &\frac{\gamma_1}{1 - \lambda_1}(1 - 2\lambda_1^k + \lambda_1^{k+1}) + \frac{\gamma_2}{1 - \lambda_2}(1 - 2\lambda_2^k + \lambda_2^{k+1}) \\ &- \frac{\mu}{2}(k-1) + a_0. \end{aligned}$$

We assert that

$$\liminf_{r \to \infty} T(r, f_1)/T(r, f_0) \geq 1 \quad \text{and} \quad \liminf_{r \to \infty} T(r, g_1)/T(r, g_0) \geq 1. \tag{78}$$

To show this, we assume that

(79) $$\liminf_{r\to\infty} T(r, f_1)/T(r, f_0) = \alpha < 1.$$

For any $\epsilon > 0$ such that $\alpha + \epsilon < 1$, there exists a sequence $\{r_n\} = \{r_n(\epsilon)\}$ satisfying

(80) $$T(r_n, f_1) \le (\alpha + \epsilon)T(r_n, f_0) \quad \text{and} \quad T(r_n, f_1) \ge (\alpha - \epsilon)T(r_n, f_0),$$

for $n \ge n_0(\epsilon)$. Later we choose a suitable ϵ. From (66),

$$\begin{aligned} T(r_n, h_0) = T(r_n, g_0) &= (T(r_n, f_0) + T(r_n, f_1))/2 + O(\log r_n) \\ &\le (T(r_n, f_0) + (\alpha + \epsilon)T(r_n, f_0))/2 + O(\log r_n) \\ &= (1 + \alpha + \epsilon)T(r_n, f_0)/2 + O(\log r_n). \end{aligned}$$

Similarly, we have

$$T(r_n, h_0) \ge (1 + \alpha - \epsilon)T(r_n, f_0)/2 + O(\log r_n).$$

We now set

$$a_0 = (1 + \alpha + \epsilon)/2, \quad b_0 = (1 + \alpha - \epsilon)/2, \quad a_1 = \alpha + \epsilon, \quad \text{and} \quad b_1 = \alpha - \epsilon.$$

We compute μ, ν, γ_1 and γ_2 concretely under our assumptions. We have $\mu = c_0 - c_1 = (a_0 + b_0) - (a_1 + b_1) = 1 - \alpha$, and $\nu = c_1 - 2c_0 = -2$. From (74),

$$a_2 = 2a_1 + a_0 + \mu + \nu = (3/2)\alpha + (5/2)\epsilon - 1/2.$$

On the other hand, from (77),

$$\begin{aligned} a_1 &= \gamma_1 + \gamma_2 + \alpha + \epsilon/2 \\ a_2 &= (1 + \lambda_1)\gamma_1 + (1 + \lambda_2)\gamma_2 + (3/2)\alpha + (1/2)\epsilon - 1/2. \end{aligned}$$

Hence we have

$$\begin{cases} \gamma_1 + \gamma_2 = \epsilon/2 \\ (1 + \lambda_1)\gamma_1 + (1 + \lambda_2)\gamma_2 = 2\epsilon. \end{cases}$$

Since $\lambda_1 = 1 + \sqrt{2}$ and $\lambda_2 = 1 - \sqrt{2}$, we obtain

$$\gamma_1 = ((1 + \sqrt{2})/4)\epsilon, \quad \gamma_2 = ((1 - \sqrt{2})/4)\epsilon.$$

Hence we can write

(81) $$\begin{aligned} a_k = &(\alpha - 1)k + (1 + \alpha + \epsilon)/2 \\ &+ \epsilon\left(\left(\frac{1 + \sqrt{2}}{4}\right)\frac{(1 + \sqrt{2})^k - 1}{\sqrt{2}} + \left(\frac{1 - \sqrt{2}}{4}\right)\frac{1 - (1 - \sqrt{2})^k}{\sqrt{2}}\right). \end{aligned}$$

Since we assume that $\alpha < 1$, we can take $k = k(\alpha)$ so large that $(\alpha-1)k+1 < 0$. Once we find a such k, we fix it. Then we choose ϵ so small that $a_k < 0$. For this ϵ, there exists $\{r_n\} = \{r_n(\epsilon)\}$ satisfying (80), in particular,

$$T(r_n, h_j) \leq a_k T(r_n, f_0) + O(\log r_n). \tag{82}$$

We observe the term $O(\log r_n)$ in (82). Write this term $\psi(\log r_n)$. Then function $\psi(x)$ in x depends on k. However, it is independent of ϵ. Since h_0 is transcendental and $a_k < 0$, the right hand side of (82) is negative for sufficiently large n, a contradiction. This gives the first inequality in (78). On the other hand, we consider a sequence of functions

$$h_j^*(z) = \begin{cases} f_j(z), & \text{if } j \text{ is even} \\ g_j(z), & \text{if } j \text{ is odd} \end{cases}$$

instead of $h_j(z)$ above. Then we obtain the second inequality in (78) by similar arguments. Hence the assertion (78) follows. It follows from (66) and the first inequality in (78) that

$$\liminf_{r\to\infty} T(r, g_0)/T(r, f_0) \geq 1. \tag{83}$$

We recall the remark that we posed after the definitions of $\{f_j(z)\}$ and $\{g_j(z)\}$, in particular,

$$T(r, g_0) + T(r, g_1) = 2T(r, f_0) + O(\log r).$$

From this and the second inequality in (78), we have

$$\liminf_{r\to\infty} T(r, f_0)/T(r, g_0) \geq 1,$$

and hence

$$\limsup_{r\to\infty} T(r, g_0)/T(r, f_0) \leq 1. \tag{84}$$

Hence we see that $\lim_{r\to\infty} T(r, f_0)/T(r, g_0) = 1$. This implies that $T(r, f_0) = (1 + o(1))T(r, g_0)$ as $r \to \infty$, which gives (28). We finally comment that the case $h_j(z) = h_i(z)$ for some $j \neq i$ is included in our arguments. We have thus proved (c). □

4. Examples

Finally we state some examples in this section. As mentioned in the statement in Theorem 2.2, a condition that gives $\#\mathfrak{S}(A) \geq 3$ is obtained and in Remark 2.6, a condition that gives $\#\mathfrak{S}(A) = 0$ is obtained.

A natural question arises: Under what conditions does $\#\mathfrak{S}(A) = 2$ occur?

We shall give some examples of A in (1) for which $\#\mathfrak{S}(A) = 2$ and an example for (23) having the property $\#\mathfrak{T}(A) = 4$.

Example 4.1 $\mathfrak{S}(1/4z) = \{\cosh\sqrt{z}, -\cosh\sqrt{z}\}$.

In fact, it is easy to see that $\cosh\sqrt{z}$ and $-\cosh\sqrt{z}$ are transcendental entire solutions of the differential equation

$$(f')^2 = (1/(4z))(f^2 - 1).$$

It follows from Theorem 2.2 (i) that there is no other solution to the equation above.

Similarly let $p(z)$ be a polynomial with simple zeros only. Then,

$$\begin{aligned}
&\mathfrak{S}(p(p')^2) = \{\pm\cosh((2/3)p^{3/2})\},\\
&\mathfrak{S}((p')^2/p) = \{\pm\cosh 2p^{1/2}\},\\
&\mathfrak{S}(z^2-1) = \{\pm\cosh(z\sqrt{z^2-1} - \log(z+\sqrt{z^2-1}))\}
\end{aligned}$$

Example 4.2 The equation

$$(f')^2 = (1/(4z))(4f^3 - \tilde{g}_2 f - \tilde{g}_3)$$

possesses a solution $\wp(\sqrt{z})$, where $\wp$ is Weierstrass' elliptic function satisfying (22). Clearly $\wp(\sqrt{z}+c)$, $c \neq 0 \in \mathbb{C}$ is not meromorphic and hence $\#\mathfrak{T}(1/4z) = 4$.

Acknowledgements We would like to thank Professor Kazunari Sawada for his valuable suggestions. We also would like to thank the referee for his/her valuable comments to improve the paper.

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