REMARKS ON DEFICIENCIES FOR MEROMORPHIC SCHRÖDER FUNCTIONS

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1. Introduction

Let $f(z)$ be a meromorphic function. Throughout this note "meromorphic" means "meromorphic in the whole complex plane $\mathbb{C}$". We use the notations of the Nevanlinna theory, $m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, $T(r, f)$, and for $\alpha \in \mathbb{C}$, $m(r, \alpha; f), N(r, \alpha; f), \overline{N}(r, \alpha; f)$ etc., see e.g. [7], [6]. We call $\alpha \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ *Picard exceptional value* if $f(z) \neq \alpha$ for any $z \in \mathbb{C}$. It is said to be a *Nevanlinna deficiency* and said to be a *Valiron deficiency* if

$$\liminf_{r\to\infty} \frac{m(r, \alpha; f)}{T(r, f)} > 0 \quad \text{and} \quad \limsup_{r\to\infty} \frac{m(r, \alpha; f)}{T(r, f)} > 0,$$

respectively. For a meromorphic function $f(z)$, we denote by $E_{\mathrm{P}}(f)$, $E_{\mathrm{N}}(f)$ and $E_{\mathrm{V}}(f)$ the set of Picard exceptional values, Nevanlinna deficiencies and Valiron deficiencies. Let $R(z)$ be a rational function of degree at least two. The set $E(R)$ of exceptional values of $R(z)$ consists of those $a \in \hat{\mathbb{C}}$ such that the equation $R^{\circ n}(z) = a$, $n \in \mathbb{N}$ have in totally a finite number of roots, where $R^{\circ n}(z)$ denotes the n-th iteration of $R(z)$. It is known that for any meromorphic solution $f(z)$ of the Schröder equation $f(sz) = R(f(z))$, where s is a complex number with $|s| > 1$, it holds

$$E(R) = E_{\mathrm{P}}(f) = E_{\mathrm{N}}(f) = E_{\mathrm{V}}(f), \tag{1.1}$$

see, [1], [10]. We treated the non-autonomous case in [2]. Namely for a fixed complex number s with $|s| > 1$, the following functional equation was considered

$$f(sz) = R(z, f(z)), \tag{1.2}$$

where $R(z, w)$ is a rational function in z and w with $\deg_w[R(z, w)] = d \geq 2$.

We continue to study the functional equation (1.2) and give two propositions in this note. We assume that $R(z, w)$ is holomorphic at $(0, 0)$

$$R(z, w) = \sum_{j,k=0}^{\infty} \alpha_{j,k} z^j w^k, \quad |z| < \delta, \ |w| < \eta.$$

The equation (1.2) admits a solution supposed that $w = R(0, w)$ has a finite root γ_0 and further that $s^n - \alpha_{0,1} \neq 0$ for any $n \geq 1$ if $\alpha_{1,0} \neq 0$, and $s - \alpha_{0,1} = 0, s^n - \alpha_{0,1} \neq 0$ for any $n \geq 2$ if $\alpha_{1,0} = 0$, see [9, p.152]. Note that in the autonomous case every meromorphic solution is transcendental, however in the non-autonomous case it does not hold in general.

We assume that (1.2) has a transcendental meromorphic solution $f(z)$. Then we have that the growth order $\rho = \rho(f)$ is equal to $\log d/\log|s|$ and the Nevanlinna characteristic function $T(r,f)$ satisfies

$$K_1 r^\rho \le T(r,f) \le K_2 r^\rho, \tag{1.3}$$

for some constants K_1 and K_2, see e.g. [9, p. 160].

We considered the question whether (1.1) would hold for a transcendental meromorphic solution of (1.2), and showed that $E_{\rm N}(f) = E_{\rm V}(f)$ does not always hold in the non-autonomous case in [2]. That is to say, we proved that *there exists a transcendental meromorphic solution of a non-autonomous equation of the form* (1.2) *satisfying* $E_N(f) \subsetneq E_V(f)$. We here show that $E_{\rm P}(f) = E_{\rm N}(f)$ does not always hold for a non-autonomous equation of the form (1.2), namely

Proposition 1. *There exists a transcendental meromorphic solution of a non-autonomous equation of the form* (1.2) *satisfying* $E_P(f) \subsetneq E_N(f)$.

We will prove Proposition 1 in Section 2.

We considered a generalization of Eremenko and Sodin's result changing a value to an algebraic function in [2]. Let $a(z)$ be an algebraic function defined by an irreducible polynomial in w with rational function coefficients

$$H(z,w) = w^p + \cdots + h_1(z)w + h_0(z) = 0, \quad h_k(z) \in \mathbb{C}(z),\ 0 \le k \le p-1. \tag{1.4}$$

Then the proximity function $m(r,a;f)$ of $f(z)$ to $a(z)$ is defined by

$$m(r,a;f) = \frac{1}{p} m(r,0;H(z,f(z))). \tag{1.5}$$

We call z_0 is a $a(z)$-point of $f(z)$ if $H(z_0, f(z_0)) = 0$. Denote by $n(r,a;f)$ the number of zeros of $H(z,f(z))$ in $|z| \le r$, divided by p, and define the counting function $N(r,a;f)$ by

$$\begin{aligned} N(r,a;f) = & \int_0^r \frac{n(t,a,f) - n(0,a,f)}{t}\,dt + n(0,a;f)\log r \\ = & \frac{1}{p}\int_0^r \frac{n(t,0;H(z,f(z))) - n(0,0;H(z,f(z)))}{t}\,dt \\ & + n(0,0;H(z,f(z)))\log r. \end{aligned} \tag{1.6}$$

Then we have $T(r,H(z,f(z)) = pT(r,f) + O(\log r)$, and from (1.5) and (1.6), $T(r,f) = m(r,a;f) + N(r,a;f) + O(\log r)$. An algebraic function $a(z)$, defined by (1.4), is said to be *a Picard exceptional function* for $f(z)$, if $H(z,f(z))$ has no zeros, i.e. $N(r,a;f)=0$. Put

$$\liminf_{r\to\infty}\frac{m(r,a;f)}{T(r,f)}=1-\limsup_{r\to\infty}\frac{N(r,a;f)}{T(r,f)}=\delta(a;f),$$
$$\limsup_{r\to\infty}\frac{m(r,a;f)}{T(r,f)}=1-\liminf_{r\to\infty}\frac{N(r,a;f)}{T(r,f)}=\Delta(a;f).$$

When $\delta(a;f)>0$, $a(z)$ is said to be a *Nevanlinna deficient function* for $f(z)$, and when $\Delta(a;f)>0$, $a(z)$ is said to be a *Valiron deficient function* for $f(z)$. We denote by $E^*_{\mathrm{P}}(f)$, $E^*_{\mathrm{N}}(f)$ and $E^*_{\mathrm{V}}(f)$ the set of Picard exceptional functions, Nevanlinna deficient functions and Valiron deficient functions. Further we call $a(z)$ a *totally ramified Picard function* if $\overline{N}(r,a;f)=o(T(r,f))$ as $r\to\infty$ including Picard functions. Denote by $E^*_{\overline{\mathrm{P}}}(f)$ the set of totally ramified Picard functions. We write $E_{\overline{\mathrm{P}}}(f)$ the set of totally ramified Picard exceptional values α satisfying $\overline{N}(r,\alpha;f)=o(T(r,f))$ as $r\to\infty$. By definition we have $E^*_{\mathrm{P}}(f)\subset E^*_{\overline{\mathrm{P}}}(f)$ and $E_{\overline{\mathrm{P}}}(f)\subset E^*_{\overline{\mathrm{P}}}(f)$.

We show in [2] that *a transcendental meromorphic solution $f(z)$ of* (1.2) *has no Valiron deficient function other than totally ramified Picard functions. That is to say, $m(r,a;f)=o(T(r,f))$, $r\to\infty$, if $a(z)$ is not a totally ramified Picard exceptional function.* This implies that

$$E^*_{\mathrm{P}}(f)\subset E^*_{\mathrm{N}}(f)\subset E^*_{\mathrm{V}}(f)\subset E^*_{\overline{P}}(f). \tag{1.7}$$

In connection with (1.1) and (1.7), we consider a question whether it would be satisfied that $E(R)=\cdots=E_{\mathrm{V}}(f)=E_{\overline{P}}(f)$ in the autonomous case.

Proposition 2. *For any meromorphic solution $f(z)$ of the Schröder equation $f(sz)=R(f(z))$, where $|s|>1$, we have*

$$E(R)=E_P(f)=E_N(f)=E_V(f)=E_{\overline{P}}(f). \tag{1.8}$$

For the proof of Propositon 2, we need some notations and refer known results. Let $f(z)$ be a meromorphic function. Let $d_{\omega_0}=\{z\ ;\ \arg[z]=\omega_0\}$ be a ray, and α be a positive number. Define a sector

$$\Omega(\omega_0,\alpha)=\{z\ ;\ |\arg[z]-\omega_0|<\alpha\}. \tag{1.9}$$

For any $a\in\hat{\mathbb{C}}$, write zeros of $f(z)-a$ (or of $1/f(z)$ when $a=\infty$) in $\Omega(\omega_0,\alpha)$ as $z_n(a,\omega_0;\alpha)$, $n=0,1,\cdots$, multiple zeros counted only once. A ray d_{ω_0} is called a *Borel direction* or a *Borel ray* for $f(z)$ if for any $\alpha>0$,

$$\sum_{n=0}^{\infty}\frac{1}{|z_n(a,\omega_0;\alpha)|^{\rho(f)-\epsilon}}=\infty\quad\text{for any }\epsilon>0 \tag{1.10}$$

with two possible exceptions of $a\in\hat{\mathbb{C}}$. Any meromorphic function of positive finite order admits Borel directions [8, p.273, Theorem VII.6]. A ray d_{ω_0} is called a *Julia* *direction* or a *Julia ray* of $f(z)$ if the function $f(z)$ takes any value $a \in \hat{\mathbb{C}}$, except two possible exceptional values, infinitely often in $\Omega(\omega_0, \alpha)$ for any $\alpha > 0$, see e.g. [5]. We call d_{ω_0} an *s-Julia direction* of $f(z)$ if the function $f(z)$ takes any value $a \in \hat{\mathbb{C}}$, except Picard exceptional values infinitely often in $\Omega(\omega_0, \alpha)$ for any $\alpha > 0$. A Borel direction of $f(z)$, with $0 < \rho(f) < \infty$, is of course a Julia direction, while the converse need not be true. Further we call d_{ω_0} is an *s-Borel direction* if (1.9) holds for any a, without exception, supposed a is not Picard exceptional value for $f(z)$. Moreover, if the left hand side of (1.9) diverges also for $\epsilon = 0$, we speak of s-Borel direction *of divergence type*, following Valiron [9, p.458]. We denote it as *sd-Borel direction*. In [3], [4], we investigate sd-Borel directions of meromorphic solutions of the Schröder equations. Write $s = |s|e^{2\pi i\lambda}$. We assume that (1.2) is of autonomous equation below. We showed [3] that *when* $\lambda \notin \mathbb{Q}$*, any meromorphic solution of* (1.2) *admits any direction as sd-Borel.* For the case $\lambda \in \mathbb{Q}$, some directions are not Julia (Borel) direction in general. We also showed that *when* $\lambda \in \mathbb{Q}$*, any Julia direction of* $f(z)$ *is s-Julia direction.* We proved [4] that *for a meromorphic solution* $f(z)$ *of* (1.2)*, any s-Julia direction is also an sd-Borel direction.* Combining these results, we have the following

Theorem A. *Suppose that* (1.2) *is autonomous. Then any meromorphic solution of* (1.2) *admits at least one direction as sd-Borel.*

We will give proof of Proposition 2 in Section 2.

2. Proofs of Propositions 1 and 2

Proof of Proposition 1. We consider a non-autonomous equation

$$f(sz) = \frac{1+z}{(1-z)^2} f(z)^2, \quad |s| > 2,$$

which admits a meromorphic solution

$$f(z) = \prod_{n=1}^{\infty} \frac{(1+z/s^n)^{2^{n-1}}}{(1-z/s^n)^{2^n}}.$$

We see that $n(r, 0; f) = 2^n - 1$ and $n(r, \infty; f) = 2^{n+1} - 1$ for $|s|^n \le r < |s|^{n+1}$. Hence we have

$$N(r, 0; f) = \sum_{k=1}^{n-1} (2^k - 1) \log |s| + (2^n - 1) \log \frac{r}{|s|^n},$$

$$N(r, \infty; f) = \sum_{k=1}^{n-1} (2^{k+1} - 1) \log |s| + (2^{n+1} - 1) \log \frac{r}{|s|^n},$$

which yields that

$$N(r,\infty;f) = 2N(r,0;f) + (n-1)\log|s| + \log\frac{r}{|s|^n}, \quad |s|^n \le r < |s|^{n+1}. \tag{2.1}$$

Assume that $\delta(0,f) = 0$. Then there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} N(t_n,0;f)/T(t_n,f) = 1$. It follows from (1.3) that $(n-1)\log|s| + \log r/|s|^n = o(T(r,f))$ as $r \to \infty$. Thus from (2.1), we get

$$\lim_{n\to\infty} N(t_n,\infty;f)/T(t_n,f) = 2,$$

a contradiction. Hence 0 must be a Nevanlinna deficient value. Clearly 0 is not Picard exceptional value, since $f(z)$ has infinitely many zeros. Therefore, we have proved Proposition 1. $\square$

Proof of Proposition 2. Suppose that there exists a value a such that $a \notin E_{\mathrm{V}}(f)$ and $a \in E_{\overline{\mathrm{P}}}(f)$. By Theorem A, there exists an sd-Borel direction d_{ω_0} of $f(z)$. Hence in a sector $\Omega(\omega_0,\alpha)$, $\alpha > 0$, $f(z)$ admits any value satisfying (1.10) with at most two exceptions which must be Picard exceptional values of $f(z)$. Since $a \notin E_{\mathrm{V}}(f)$, and hence $a \notin E_{\mathrm{P}}(f)$, we see that $\lim_{r\to\infty} \overline{N}(r,a,f,\Omega(\omega_0,\alpha))/T(r,f) > 0$, where $\overline{N}(r,a,f,\Omega(\omega_0,\alpha))$ is the counting function which counts distinct a-points in the sector $\Omega(\omega_0,\alpha)$. Clearly we have $\overline{N}(r,a;f) \ge \overline{N}(r,a,f,\Omega_\alpha)$, and hence $a \notin E_{\overline{\mathrm{P}}}(f)$ which yields a contradiction. $\square$

References

[1] A. É. Eremenko and M. L. Sodin, Iteration of rational functions and the distribution of the values of the Poincaré functions, J. Soviet Math. **58** (1992), 504–509.
[2] K. Ishizaki and N. Yanagihara, Deficiency for meromorphic solutions of the Schröder equations, Complex Variables Theory Appl., **49**, (2004), 539–548.
[3] K. Ishizaki and N. Yanagihara, Borel and Julia directions of meromorphic Schröder functions, to appear in Math. Proc. Camb. Phil. Soc.
[4] K. Ishizaki and N. Yanagihara, Borel and Julia directions of meromorphic Schröder functions II, Preprint.
[5] G. Julia, Sur quelques propriétés nouvelles des fonctions entières ou méromorphes, Ann. Sci. École Norm. Sup. **36** (1919), 93–125.
[6] I. Laine, Nevanlinna Theory and Complex Differential Equations, Walter de Gruyter, Berlin 1993.
[7] R. Nevanlinna, Analytic Functions, Springer-Verlag, New York-Berlin 1970.
[8] M. Tsuji, Potential Theory in Modern Function Theory, Maruzen, Tokyo 1959.
[9] G. Valiron, Fonctions Analytiques, Press. Univ. de France, Paris 1954.
[10] N. Yanagihara, Exceptional values for meromorphic solutions of some difference equations, J. Math. Soc. Japan, **34** (1982), 489–499.

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