BOREL AND JULIA DIRECTIONS OF MEROMORPHIC SCHRÖDER FUNCTIONS

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Abstract. Meromorphic solutions of the Schröder equation $f(sz) = R(f(z))$ are studied, where $|s| > 1$ and $R(w)$ is a rational function with $\deg[R] \geq 2$. We will show that, if $\arg[s] \notin 2\pi\mathbb{Q}$, then $f(z)$ has any direction as Borel, and besides, without exceptional values other than Picard values, which depend on $R(w)$. Further the case $\arg[s] \in 2\pi\mathbb{Q}$ is also considered. We investigate the relation between Julia directions of $f(z)$ and the Julia set of $R(w)$.

1. Introduction

Let $f(z)$ be a transcendental meromorphic function in the complex plane $\mathbb{C}$. We call $a \in \mathbb{C} \cup \{\infty\} = \hat{\mathbb{C}}$ is a *Picard exceptional value*, if $f(z)$ does not have any zeros of $f(z) - a$, or zero of $1/f(z)$ when $a = \infty$. Let $d_{\omega_0} = \{z \,;\, \arg[z] = \omega_0\}$ be a ray, and α be a positive number. Define a sector

$$\Omega(\omega_0, \alpha) = \{z \,;\, |\arg[z] - \omega_0| < \alpha\}. \tag{1.1}$$

A ray d_{ω_0} is called a *Julia direction* or a *Julia ray* of $f(z)$ if the function $f(z)$ takes any value $a \in \hat{\mathbb{C}}$, except two possible exceptional values, infinitely often in $\Omega(\omega_0, \alpha)$ for any $\alpha > 0$, see e.g. [4]. We call d_{ω_0} an *s-Julia direction* of $f(z)$ if the function $f(z)$ takes any value $a \in \hat{\mathbb{C}}$, except Picard exceptional values infinitely often in $\Omega(\omega_0, \alpha)$ for any $\alpha > 0$. Denote by $\rho = \rho(f)$ the order of growth of $f(z)$. For any $a \in \hat{\mathbb{C}}$, write zeros of $f(z) - a$ (or of $1/f(z)$ when $a = \infty$) in $\Omega(\omega_0, \alpha)$ as $z_n(a, \omega_0; \alpha)$, $n = 0, 1, \cdots$, multiple zeros counted only once. A ray d_{ω_0} is called a *Borel direction* or a *Borel ray* for $f(z)$ if for any $\alpha > 0$,

$$\sum_{n=0}^{\infty} \frac{1}{|z_n(a, \omega_0; \alpha)|^{\rho(f)-\epsilon}} = \infty \quad \text{for any } \epsilon > 0 \tag{1.2}$$

with two possible exceptions of $a \in \hat{\mathbb{C}}$. Any meromorphic function of positive finite order admits Borel directions [9, p.273, Theorem VII.6]. A Borel direction of $f(z)$,

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with $0 < \rho(f) < \infty$, is of course a Julia direction, while the converse need not be true. Further we call d_{ω_0} is a *s-Borel direction* if (1.2) holds for any a, without exception, supposed a is not Picard exceptional value for $f(z)$. Moreover, if the left hand side of (1.2) diverges also for $\epsilon = 0$, we speak of s-Borel direction *of divergence type*, following Valiron [10, p.458]. We denote it as *sd-Borel direction*.

Let s be a complex number such that $|s| > 1$. Write $s = |s|e^{2\pi\lambda i}$, $\lambda \in [0, 1)$. We consider the Schröder equation

$$f(sz) = R(f(z)) = P(f(z))/Q(f(z)), \tag{1.3}$$

where $P(w) = a_0 + a_1 w + \cdots + a_p w^p$ and $Q(w) = b_0 + b_1 w + \cdots + b_q w^q$, $a_j, b_j \in \mathbb{C}$, $a_p b_q \neq 0$. We assume that $P(w)$ and $Q(w)$ are relatively prime polynomials, and assume $m = \deg[R(w)] = \max(p, q) \geq 2$.

It is well known that under some conditions (1.3) possesses a transcendental meromorphic solution. We call this solution a Schröder function. It is also known that the growth order of the Schröder function is given by $\rho = \log m / \log |s| > 0$. See, for example, [3], [11].

First we state a result in the case $\lambda \notin \mathbb{Q}$.

Theorem 1. *Suppose $\lambda \notin \mathbb{Q}$. Then a Schröder function $f(z)$ of* (1.3) *admits any direction as sd-Borel.*

Also when $\lambda \in \mathbb{Q}$, there may occur similar situation. For example, $\wp$ function satisfies

$$\wp(2z) = R(\wp(z)) = \frac{1}{4} \cdot \frac{(6\wp(z)^2 - g_2/2)^2}{4\wp(z)^3 - g_2\wp(z) - g_3} - 2\wp(z),$$

and admits any direction as sd-Borel, in which the Julia set $\mathcal{J}_R$ of $R(w)$ coincides with $\hat{\mathbb{C}}$ [8, p.32].

Secondly we consider the case $\lambda \in \mathbb{Q}$. We investigate the relation between Julia directions of $f(z)$ and the Julia set of $R(w)$. Write $\lambda = \mu/\nu$, where μ and $\nu \neq 0$ are integers. Then by considering $R^{2\nu}(w)$ for $R(w)$, we can suppose that $s > 1$, since $\mathcal{J}_{R^{2\nu}} = \mathcal{J}_R$ by [8, p.28, Theorem 3].

Theorem 2. *Suppose $s > 1$ in* (1.3). *Let $f(z)$ be a Schröder function of* (1.3). *A ray d_{ω_0} is s-Julia direction if and only if $f(d_{\omega_0}) \cap \mathcal{J}_R \neq \emptyset$. Further, any Julia direction for $f(z)$ is s-Julia.*

The example of $\wp(z)$, stated above, shows that it may occur $f(d_{\omega_0}) \subsetneq \mathcal{J}_R$. Examples for $f(d_{\omega_0}) = \mathcal{J}_R$ and $f(d_{\omega_0}) \supsetneq \mathcal{J}_R$ will be given later in Section 6, see Examples 3 and 4.

Suppose $R(0) = 0$ and $R'(0) = s$ in (1.3). Then there exists a Schröder function of (1.3) such that $f(0) = 0$ and $f'(0) = 1$, see e.g. [8]. Then the origin is a repelling fixed point of $R(w)$ which is contained in $\mathcal{J}_R$. Let $\widehat{\mathcal{J}_R}(0)$ be a limit set of $\arg[w], w \in \mathcal{J}_R$ at $w = 0$, [8, p.125], i.e.,

$$\widehat{\mathcal{J}_R}(0) = \bigcap_{\epsilon>0} \overline{\{\arg[w]\ ;\ w \in \mathcal{J}_R,\ 0 < |w| < \epsilon\}}. \tag{1.4}$$

For the Schröder function of (1.3), we define

$$\mathbb{J}_f = \{\omega\ ;\ d_\omega \text{ is a Julia direction of } f(z)\}. \tag{1.5}$$

Obviously $\mathbb{J}_f$ is a closed set.

Theorem 3. *Suppose that $R(0) = 0$ and $R'(0) = s$ in (1.3). Let $f(z)$ be a Schröder function with $f(0) = 0$ and $f'(0) = 1$. Then, the sets $\widehat{\mathcal{J}_R}(0)$ and $\mathbb{J}_f$ coincide.*

We mention properties of Schröder functions, in particular, their exceptional values in Section 2. The characteristic function in a sector due to Tsuji is studied in Section 3. We prove Theorem 1 in Section 4. In Section 5, we show Theorems 2 and 3. We give examples in Section 6.

2. Exceptional values

Let $R(w) = P(w)/Q(w)$ be a rational function with $\deg[R] = m \geq 2$, where $P(w)$ and $Q(w)$ are relatively prime polynomials. When there exists a value $b \in \hat{\mathbb{C}}$ such that $b = R(w)$ implies $w = b$ with multiplicity m, we call such value b a maximally fixed value of $R(w)$. When there are b and $c \in \hat{\mathbb{C}},\ b \neq c$, such that $b = R(w)$ implies $w = c$ and $c = R(w)$ implies $w = b$, with multiplicity m respectively, we call such a pair of values b and c a maximally fixed pair of $R(w)$. See [13].

We recall elementary properties of the Schröder equation, when $R(w)$ admits a maximally fixed value or a maximally fixed pair.

Case 1. If there is a maximally fixed value $b \in \hat{\mathbb{C}}$, then putting $1/(f(z) - b)$ as $f(z)$, we get

$$f(sz) = \tilde{P}(f(z)) = a_0 + a_1 f(z) + \cdots + a_p f(z)^p. \tag{2.1}$$

Case 2. If there are two maximally fixed values b and $c \in \hat{\mathbb{C}},\ b \neq c$, then putting $A(f(z) - b)/(f(z) - c)$ as $f(z)$, with a constant A, we get

$$f(sz) = f(z)^p. \tag{2.2}$$

Case 3. If there exists a maximally fixed pair b and $c \in \hat{\mathbb{C}},\ b \neq c$, then putting similarly to the case 2, we get

$$f(sz) = 1/f(z)^q. \tag{2.3}$$

We have the following proposition.

Proposition 4. *A value b is a Picard value for the Schröder function of* (1.3) *if and only if b is a maximally fixed value of $R(w)$ in* (1.3) *or there exists a value $c \neq b$ such that b and c construct a maximally fixed pair of $R(w)$.*

It seems that Proposition 4 is a known fact. For the convenience for the reader, we give a proof below.

Proof of Proposition 4. Suppose that $R(w)$ admits a maximally fixed value b. If the case 1 above occurs, then we get (2.1). It is clear that any solution of (2.1) has no poles. If the case 2 above occurs with another maximally fixed value c, we get (2.2). The only solution of (2.2) is $f(z) = \exp[Cz^\kappa], C \in \mathbb{C}$, where $\kappa \in \mathbb{N}$ and $s^\kappa = p$. Obviously, any solution of (2.1) has no poles nor zeros. This shows that b is a Picard value of $f(z)$. Next we suppose that $R(w)$ admits a maximally fixed pair b and c, say the case 3 occurs. Similarly the only solution of (2.3) is $f(z) = \exp[Cz^\kappa]$, with $s^\kappa = -q$, which implies that b and c are Picard values of $f(z)$.

Conversely suppose that b is a Picard value of $f(z)$. Thus $f(sz)-b$ has no zeros. From (1.3), we can write $f(sz)-b = R(f(z))-b = A(f(z)-b)^{k_1}(f(z)-c)^{k_2}/Q(w)$, $k_1+k_2 = d$ for some $c \in \mathbb{C}$, $c \neq b$, where A is a constant, k_1 and k_2 are non-negative integers, since the number of Picard values is at most two. If $k_2 = 0$, then b is a maximally fixed value of $R(w)$. It suffices to consider the case $k_2 \neq 0$. In this case, c is also a Picard value for $f(z)$ and $R(c) = b$. Thus we can write $f(sz) - c = R(f(z)) - c = \tilde{A}(f(z) - b)^{\tilde{k}_1}(f(z) - c)^{\tilde{k}_2}/Q(w)$, $\tilde{k}_1 + \tilde{k}_2 = d$ where $\tilde{A}$ is a constant, $\tilde{k}_1$ and $\tilde{k}_2$ are non-negative integers. If $\tilde{k}_2 \neq 0$, then $R(c) = c$, a contradiction. If $\tilde{k}_2 = 0$, then $\tilde{k}_1 \neq 0$ and $R(b) = c$. Thus we have $k_1 = 0$. In fact, if we assume the contrary, say $k_1 \neq 0$, then $R(b) = b$, a contradiction. This concludes that a pair of values b and c is a maximally fixed pair of $R(w)$. □

The set $E(R)$ of exceptional values of a rational function $R(z)$ consists of those $a \in \mathbb{C} \cup \{\infty\}$ such that the equation $R^n(z) = a$, $n \in \mathbb{N}$ have in totality a finite number of roots, where $R^n(z)$ denotes the n-th iteration of $R(z)$. The set $E(R)$ coincides with the set of maximally fixed values of $R(w)$ and values of a pair of maximally fixed values of $R(w)$. By Proposition 4, $E(R)$ coincides with the set of Picard exceptional values of the corresponding Schröder function $f(z)$. At the end of this section, we state the known proposition, see e.g. [5, §14].

Proposition 5. *Let $R(w)$ be a rational function, and let V be an open set such that $V \cap \mathcal{J}_R \neq \emptyset$. Then, for any compact set $K \subset \hat{\mathbb{C}} \setminus E(R)$, there is an integer n_0 such that $R^n(V) \supset K$ for any $n \geq n_0$.*

3. Characteristic functions in angular domains

There are several ways in defining the characteristic function for functions meromorphic in a sector, see [2, Chapter I §5], [6], [12]. We follow here Tsuji [9, p. 272].

Let α be a positive number, and $\Omega(\omega_0, \alpha)$ a sector defined in (1.1). Write $\Omega_\alpha(r) = \Omega(\omega_0, \alpha) \cap \{|z| \leq r\}$. For a meromorphic function w in $\Omega(\omega_0, \alpha)$, we define the characteristic function

$$T(r; \Omega_\alpha; w) = \int_0^r \frac{S(t, \Omega_\alpha; w)}{t} dt, \tag{3.1}$$

where

$$S(r; \Omega_\alpha; w) = \frac{1}{\pi} \iint_{\Omega_\alpha(r)} \left(\frac{|w'(re^{i\theta})|}{1 + |w(re^{i\theta})|^2} \right)^2 r dr d\theta. \tag{3.2}$$

We state an analogue for the second fundamental theorem of Nevanlinna. We denote by $\overline{n}(r, b; \Omega_\alpha; w)$ be the number of zeros of $w(z) - b$ (when $b = \infty$, we take $1/w$ for $w - b$), contained in $\Omega_\alpha(r)$, multiple zeros counted only once, independently of its multiplicity. Define

$$\overline{N}(r, b; \Omega_\alpha; w) = \int_1^r \frac{\overline{n}(r, b; \Omega_\alpha; w)}{t} dt. \tag{3.3}$$

We state an analogue for the second fundamental theorem of Nevanlinna. For any $\alpha_0 > \alpha$, we have $S(r; \Omega_\alpha; w) \leq 3 \sum_{i=1}^3 \overline{n}(2r, b_i; \Omega_{\alpha_0}; w) + O(\log r)$, see [9, p. 272, Theorem VII.3], and hence

$$T(r; \Omega_\alpha; w) \leq 3 \sum_{i=1}^3 \overline{N}(2r, b_i; \Omega_{\alpha_0}; w) + O((\log r)^2). \tag{3.4}$$

Now we study how $T(r; \Omega_\alpha; w)$ behaves under some elementary operations with respect to w, since there is no analogue of the first fundamental theorem for $T(r; \Omega_\alpha; w)$. One purpose is to obtain an analogue for the Valiron-Mohon'ko theorem.

Proposition 6. *Let $R(w)$ be a rational function, and $f(z)$ be a meromorphic function in Ω_α. Then for a constant $L > 0$, we have*

$$T(r; \Omega_\alpha; R(f)) \leq L T(r; \Omega_\alpha; f).$$

Proof of Proposition 6. Let k be a positive integer. We write $f^\#(z) = |f'(z)|/(1 + |f(z)|^2)$ in what follows. For $w = (f(z))^k$, using $(|f|^{k-1} - 1)(|f|^{k+1} - 1) = |f|^{2k} - (|f|^{k-1} + |f|^{k+1}) + 1 \geq 0$, we get

$$\begin{aligned} (f(z)^k)^\# &= k \frac{(1 + |f(z)|^2)|f(z)|^{k-1}}{1 + |f(z)|^{2k}} \cdot \frac{|f'(z)|}{1 + |f(z)|^2} \\ &= k \frac{|f(z)|^{k-1} + |f(z)|^{k+1}}{1 + |f(z)|^{2k}} \cdot \frac{|f'(z)|}{1 + |f(z)|^2} \leq k f^\#(z). \end{aligned}$$

This gives that $S(r;\Omega_\alpha;f^k)\le k^2S(r;\Omega_\alpha;f)$, and hence

$$T(r;\Omega_\alpha;f^k)=T(r;\Omega_\alpha;1/f^k)\le k^2T(r;\Omega_\alpha;f). \tag{3.5}$$

For $w=af(z)$ with $a\in\mathbb{C},\ a\ne 0$, we have $(af(z))^{\#}\le \max(|a|,1/|a|)f^{\#}(z)$. Hence we have

$$T(r;\Omega_\alpha;af)\le(\max\{|a|,1/|a|\})^2T(r;\Omega_\alpha;f). \tag{3.6}$$

Let $P_1(w)=a_0+a_1w+\cdots+a_{k'}w^{k'},\ P_2(w)=P_1(w)+a_kw^k, k'<k$, and $Q(w)=b_0+b_1w+\cdots+b_qw^q$. Suppose that $Q(w)$ and $P_2(w)$ are relatively prime. Put $w_1(z)=P_1(f(z))/Q(f(z))$ and $w_2(z)=a_kf(z)^k/Q(f(z))$. We assert that

$$T(r;\Omega_\alpha;w_1+w_2)\le L_1\{T(r;\Omega_\alpha;w_1)+T(r;\Omega_\alpha;w_2)\}, \tag{3.7}$$

with a constant L_1, which depends on w_1 and w_2. In fact, we have

$$(w_1+w_2)^{\#}\le\frac{1+|w_1|^2}{1+|w_1+w_2|^2}w_1^{\#}+\frac{1+|w_2|^2}{1+|w_1+w_2|^2}w_2^{\#}.$$

When $|w_2(z)|\ge 2|w_1(z)|$ or $|w_2(z)|\le(1/2)|w_1(z)|$, we have $|w_1+w_2|\ge(1/2)|w_2|\ge|w_1|$, or $|w_1+w_2|\ge(1/2)|w_1|\ge|w_2|$. Thus, we obtain in both cases

$$\frac{1+|w_1|^2}{1+|w_1+w_2|^2}\le 4\quad\text{and}\quad\frac{1+|w_2|^2}{1+|w_1+w_2|^2}\le 4.$$

When $(1/2)|w_1(z)|\le|w_2(z)|\le 2|w_1(z)|$, we see that $|a_k||f(z)|^k\le 2\sum_{j=0}^{k'}|a_j||f(z)|^j$. Hence $|f(z)|\le K$ for some K. Since $Q(w)$ and $P_2(w)$ have no common zeros, we have $|Q(w)|^2+|P_2(w)|^2\ge K^*>0$ with some K^*, for $|w|\le K$. Further $|Q(w)|^2\le K_1,\ |P_1(w)|^2\le K_2$ for $|w|\le K$. Hence

$$\frac{1+|w_1|^2}{1+|w_1+w_2|^2}=\frac{|Q(f(z))|^2+|P_1(f(z))|^2}{|Q(f(z))|^2+|P_2(f(z))|^2}\le\frac{K_1+K_2}{K^*}\le L_2,$$
$$\frac{1+|w_2|^2}{1+|w_1+w_2|^2}=\frac{|Q(f(z))|^2+|a_kf(z)^k|^2}{|Q(f(z))|^2+|P_2(f(z))|^2}\le\frac{K_1+|a_kK|^2}{K^*}\le L_2,$$

with some constant $L_2\ge 4$. This implies $((w_1+w_2)^{\#})^2\le 2L_2^2\big((w_1^{\#})^2+(w_2^{\#})^2\big)$, from which (3.7) follows. Applying (3.7) and (3.6) repeatedly, and using (3.5), we obtain

$$\begin{aligned}T(r;\Omega_\alpha;R(f))&\le L_3\sum_{k=0}^{p}T\left(r;\Omega_\alpha;\frac{a_kf^k}{Q(f)}\right)=L_3\sum_{k=0}^{p}T\left(r;\Omega_\alpha;\frac{Q(f)}{a_kf^k}\right)\\&\le L_4\sum_{k=0}^{\max(p,q)}\big(T(r;\Omega_\alpha;f^k)+T(r;\Omega_\alpha;1/f^k)\big)\le LT(r;\Omega_\alpha;f),\end{aligned}$$

where L_3, L_4 and L are constants. Therefore, we have proved Proposition 6. □

The inequality (3.7) and some estimates, which are satisfied in the argument in the complex plane, does not always hold in general. We give counter examples in Section 6, see Examples 1 and 2.

4. Proof of Theorem 1

Proof of Theorem 1. Let $T(r,f)$ be the characteristic of $f(z)$ in the sense of Shimizu–Ahlfors [9] p.196. The Schröder function $f(z)$ of (1.3) is known to satisfy

$$C_1 r^\rho \le T(r,f) \le C_2 r^\rho, \quad \rho = \log m/\log|s|,$$

for some constants $0 < C_1 < C_2$, see [11]. Hence we have

$$\int^\infty \frac{T(r,f)}{r^{\rho+1}} dr = \infty.$$

Dividing $\mathbb{C}$ into two sectors $\Omega^1 = \Omega(0,\pi/2)$ and $\Omega^2 = \Omega(\pi,\pi/2)$, we obtain

$$\int^\infty \frac{T(r;\Omega^j;f)}{r^{\rho+1}} dr = \infty \quad \text{for } j=1 \text{ or } j=2.$$

When, e.g., it holds for $j=1$, we divide Ω^1 into two sectors. Repeating this procedure, we get a direction d_{ω^*} such that, for $\Omega_n^* = \Omega(\omega^*, 2\pi/2^n)$, we have

$$\int^\infty \frac{T(r;\Omega_n^*;f)}{r^{\rho+1}} dr = \infty$$

for any n. Take a direction d_{ω_0} and a sector $\Omega(\omega_0,\alpha)$. Let $2\pi/2^{n_0} < \alpha/8$. There is j_0 such that $|(\omega_0 + 2\pi\lambda j_0) - \omega^*| < \alpha/8 \pmod{2\pi}$. By (1.3) we obtain, writing the j_0-th iteration of $R(w)$ as $R^{j_0}(w)$, we have $f(z) = R^{j_0}(f(s^{-j_0}z))$. Thus by Proposition 6, with some constant $L(j_0)$,

$$T(r;\Omega_{j_0}^*;f) \le L(j_0) T(|s|^{-j_0} r; \Omega_{\alpha/4}; f),$$

hence

$$\int^\infty \frac{T(r;\Omega_{\alpha/4};f)}{r^{\rho+1}} dr = \infty.$$

By (3.4), it can hold that, with α_0 $(\alpha/4 < \alpha_0 < \alpha)$,

$$\int^\infty \frac{N(r;b_i;\Omega_{\alpha_0};f)}{r^{\rho+1}} dr < \infty \tag{4.1}$$

for at most two values b_1, b_2, which proves that any direction is Borel for $f(z)$.

We show that Borel direction is s-Borel direction below. Assume that (4.1) holds for b_1 which is not Picard value for $f(z)$. Then in $\Omega(\omega_0, \alpha/4)$,

$$\sum_{n=0}^\infty \frac{1}{|z_n(b_1,\alpha/4)|^\rho} < \infty.$$

Put $\mu = \min\{\nu \in \mathbb{N}\ ;\ \nu > 0,\ |\nu\lambda| < \alpha/8 \mod 2\pi\}$. From (1.3) we obtain $f(s^\mu z) = R^\mu(f(z))$. By Proposition 4, neither the case 2 nor the case 3 in Section 2 occurs here. Hence $b_1 = R^\mu(w)$ has a root $b^* \neq b_1$, for which we would have

$$\sum_{n=0}^{\infty} \frac{1}{|z_n(b^*, \alpha)|^\rho} < \infty,$$

a contradiction, if $b^* \neq b_2$. Suppose $b^* = b_2$. Then $b^* = R^\mu(w)$ has a root $b^{**} \neq b^* = b_2$, which we can take $b^{**} \neq b_1$. Then we would have

$$\sum_{n=0}^{\infty} \frac{1}{|z_n(b^{**}, \alpha)|^\rho} < \infty,$$

a contradiction, which proves that there is no exceptional value other than Picard values. □

5. Proofs of Theorems 2 and 3

Proof of Theorem 2. We suppose that $f(d_{\omega_0}) \cap \mathcal{J}_R \neq \emptyset$. Let $z_0 \in d_{\omega_0}$ be such that $f(z_0) \in \mathcal{J}_R$ and $U = U_{\delta_0} = \{z; |z - z_0| < \delta_0\}$. Write $\alpha_0 = \sin^{-1}(\delta_0/|z_0|)$. Then $V = f(U)$ is open and $V \cap \mathcal{J}_R \neq \emptyset$. By Proposition 5, for any $w^* \in \hat{\mathbb{C}} \setminus E(R)$, there exists an intger n_0, we have $w^* \in R^n(V) = R^n(f(U)) = f(s^n U), n \geq n_0$, since $f(z)$ is the Schröder function. Therefore, $s^n U = \{s^n z; z \in U\}, n \geq n_0$, contain z_n such that $f(z_n) = w^*$. Thus $f(z)$ takes w^* infinitely often in the sector $\Omega(\omega_0, \alpha_0) = \{z; |\arg[z] - \omega_0| < \alpha_0\}$. Since δ_0 is arbitrary and hence $\alpha_0 > 0$ is arbitrary, d_{ω_0} is an s-Julia direction.

Suppose d_{ω_0} is an s-Julia direction. Let $w_0 \notin E(R)$ be a point of $\mathcal{J}_R$. Let $\alpha_n \downarrow 0$ and write $\Omega(\omega_0, \alpha_n)$ as Ω_n. Then there is $z_0(n) \in \Omega_n$ such that $f(z_0(n)) = w_0$. We can take $\ell(n) \in \mathbb{N}$ such that $s \leq |z_0|/s^{\ell(n)} \leq s^2$. For any $z \in \mathbb{C}$ and $m \in \mathbb{N}$, we have $f(z/s^m) \in R^{-m}(f(z))$ by (1.3). Thus, if we write $z_n = z_0(n)/s^{\ell(n)}$, then $z_n \in \Omega_n$ and $f(z_n) \in R^{-\ell(n)}(w_0) \subset \mathcal{J}_R$. Since $\alpha_n \downarrow 0$, we have that z_{n_k}, for a subsequence (n_k), converges to a point $z^* \in d_{\omega_0}$. Since $f(z_{n_k}) \in \mathcal{J}_R$ and $f(z_{n_k}) \to f(z^*)$, we get $f(z^*) \in \mathcal{J}_R$. Hence $f(d_{\omega_0}) \cap \mathcal{J}_R \neq \emptyset$.

We show that a Julia direction is an s-Julia direction. Let d_ω be a Julia direction. Suppose $f(z)$ would take $a \notin E(R)$ only finite times in $\Omega(\omega, \alpha)$. Since there are $b \in R^{-1}(a)$ and $c \in R^{-1}(b)$ such that $(b - a)(c - a)(c - b) \neq 0$, we see that $f(z)$ could take a, b and c only finitely many times in $\Omega(\omega, \alpha)$, which contradicts that d_ω is a Julia direction for $f(z)$. Therefore $f(z)$ has infinitely many a-points in $\Omega(\omega, \alpha)$ for any $a \notin E(R)$. □

Proof of Theorem 3. Suppose $\phi \in \widehat{\mathcal{J}_R}(0)$. There are $w_\nu \in \mathcal{J}_R$ such that $w_\nu \to 0$, $\arg[w_\nu] \to \phi$ as $\nu \to \infty$. We can take $\epsilon > 0$ and $\eta > 0$ such that $|w| < \epsilon$ is mapped into $|z| < \eta$ homeomorphically by $w = f(z)$. If $|w_\nu| < \epsilon$, then there is z_ν

with $w_\nu = f(z_\nu)$, hence $z_\nu = w_\nu(1 + O(w_\nu))$ and

$$\arg[z_\nu] = \arg[w_\nu] + \arg[(1 + O(w_\nu))] = \arg[w_\nu] + o(1), \quad \text{as } \nu \to \infty.$$

By means of Theorem 2, we see that $\omega_\nu = \arg[z_\nu] \in \mathbb{J}_f$, since $f(z_\nu) = w_\nu \in \mathcal{J}_R$. We have that $\mathbb{J}_f$ is closed. Hence $\phi = \lim_{\nu\to\infty} \omega_\nu \in \mathbb{J}_f$. Therefore $\widehat{\mathcal{J}_R}(0) \subset \mathbb{J}_f$.

Suppose $\omega \in \mathbb{J}_f$. By Theorem 2, there is a point $z_0 \in d_\omega$ such that $f(z_0) = w_0 \in \mathcal{J}_R$. Put $f(z_0/s^n) = w_0^{(n)}$. Then by (1.3), $w_0^{(n)} \in R^{-n}(w_0) \in \mathcal{J}_R$. Since $f(0) = 0$, $w_0^{(n)}$ tend to 0 as $n \to \infty$. Similarly to the case above, we can write

$$w_0^{(n)} = \frac{z_0}{s^n}(1 + O(z_0/s^n)) \quad \text{and} \quad \frac{z_0}{s^n} = w_0^{(n)}(1 + O(w_0^{(n)})),$$

and hence

$$\omega = \arg[z_0] = \arg[w_0^{(n)}] + \arg[1 + O(w_0^{(n)})] = \arg[w_0^{(n)}] + \delta_n,$$

where $\delta_n \to 0$ as $n \to \infty$. For any $\epsilon > 0$, we choose n_0 sufficiently large so that $\omega - \delta_n = \arg[w_0^{(n)}] \in \overline{\{\arg[w]\ ;\ w \in \mathcal{J}_R,\ 0 < |w| < \epsilon\}}$, $n \geq n_0$. This implies $\omega \in \widehat{\mathcal{J}_R}(0)$. Thus $\mathbb{J}_f \subset \widehat{\mathcal{J}_R}(0)$. $\square$

6. Examples

First we give an example in connection with Section 2. The following example shows that (3.7) does not hold in general.

Example 1. We consider functions $w_1(z) = e^{2z} + e^{-iz}$ and $w_2(z) = -e^{2z}$. For $0 < \epsilon < \pi/2$, we define sectors $\Omega_\epsilon^{\pm\pi/2} = \Omega(\pm\pi/2, \pi/2 - \epsilon)$, $\Omega_\epsilon^0 = \Omega(0, \epsilon)$ and $\Omega_\epsilon^\pi = \Omega(\pi, \epsilon)$. We compute

$$\begin{aligned} S(r; \Omega_\epsilon^0; w_2) &= \frac{1}{\pi} \iint_{t \leq r, |\theta| \leq \epsilon} \frac{|-2e^{2z}|^2}{(1 + |e^{2z}|^2)^2} t dt d\theta \quad (z = te^{i\theta}) \\ &= \frac{1}{\pi} \iint_{t \leq r, |\theta| \leq \epsilon} \frac{4e^{4t\cos\theta}}{(1 + e^{4t\cos\theta})^2} t dt d\theta \leq \frac{2\epsilon}{\pi} \int_0^r \frac{4e^{4t\cos\epsilon}}{(1 + e^{4t\cos\epsilon})^2} t dt = O(1), \end{aligned}$$

and hence

$$T(r; \Omega_\epsilon^0; w_2) = O(\log r). \tag{6.1}$$

We choose an ϵ sufficiently small satisfying $\sin\epsilon - 2\cos\epsilon \leq -1$. Then by similar computations, we get

$$T(r; \Omega_\epsilon^0; w_1) = O(\log r). \tag{6.2}$$

On the other hand, we assert that for $e^{-iz} = w_1 + w_2$

$$T(r; \Omega_\epsilon^0; e^{-iz}) = T(r; \Omega_\epsilon^\pi; e^{-iz}) = (1/2\pi)r + O(\log r). \tag{6.3}$$

In fact, it is well known that $T(r; e^{-iz}) = (1/\pi)r + O(1)$. It is easy to see that

$$T(r; \Omega_\epsilon^{\pi/2}; e^{-iz}) = T(r; \Omega_\epsilon^{-\pi/2}; e^{-iz}) = O(\log r),$$

which gives (6.3). It follows from (6.1), (6.2) and (6.3) that

$$T(r;\Omega^0_\epsilon;w_1+w_2)\le L\left(T(r;\Omega^0_\epsilon;w_1)+T(r;\Omega^0_\epsilon;w_2)\right)$$

does not hold for any L.

Let $f_1(z)$ and $f_2(z)$ be meromorphic functions in the complex plane. We have $T(r,f_1f_2)\le T(r,f_1)+T(r,f_2)+O(1)$. However, in the arguments in the sector, the corresponding inequality in terms of the characteristic function defined by (3.1) does not hold.

Example 2. Put $w_1(z)=e^{2z}e^{-iz}$ and $w_2(z)=e^{-2z}$. We have

$$T(r;\Omega^0_\epsilon;w_1w_2)=T(r;\Omega^0_\epsilon;e^{-iz})=(1/2\pi)r+O(\log r).$$

On the other hand, by similar computations in Example 1, we have

$$T(r;\Omega^0_\epsilon;w_1)=O(\log r)\quad\text{and}\quad T(r;\Omega^0_\epsilon;w_2)=O(\log r).$$

Hence $T(r;\Omega^0_\epsilon;w_1w_2)\le L\big(T(r;\Omega^0_\epsilon;w_1)+T(r;\Omega^0_\epsilon;w_2)\big)$ does not hold for any constant L.

Example 3. Suppose $P(w)$ be a polynomial such that $\mathcal{J}_P$ is an analytic curve or arc. Then there are only finitely many Borel (Julia) directions for the solution $f(z)$ of (2.1). In this case, we have $f(d_{\omega_0})=\mathcal{J}_P$ if d_{ω_0} is a Julia direction.

In fact, since $\mathcal{J}_P$ is an analytic curve or arc, it can contain at most countably many double points. Hence it must be a Jordan curve or arc [8, p.140 Theorem 3].

If it is a Jordan arc, then $P(w)$ with $\deg[P]=m$ is conjugate to $\pm T_m(w)$, where T_m denotes the m-th Tchebychev polynomial [8, p.143 Theorem 5]. Hence Borel directions of $f(z)$ are finite in number. For example, if $P(w)=4w+w^2$, then $f(z)=-4\sin^2(\sqrt{-z/4})$ satisfies $f(4z)=P(f(z)), f(0)=0$ and $f'(0)=1$. In this case $\mathcal{J}_P=[-4,0]$ and the only Borel direction d_π is the negative real axis. Obviously we have $f(d_\pi)=\mathcal{J}_P$.

If $\mathcal{J}_P$ is a Jordan curve, then $P(w)$ is conjugate to w^m [8, p.145 Exercise 8], hence the number of Borel directions of $f(z)$ are also finite. Thus it is easily to see that $f(d_{\omega_0})=\mathcal{J}_P$. For example, if $P(w)=2w+w^2$, then $f(z)=e^z-1$ satisfies $f(2z)=P(f(z)), f(0)=0$ and $f'(0)=1$. In this case $\mathcal{J}_P=\{|w+1|=1\}$ and the only Borel directions are $d_{\pi/2}, d_{3\pi/2}$, i.e. positive and negative imaginary axes. It is obvious that $f(d_{\pi/2})=f(d_{3\pi/2})=\mathcal{J}_P$.

Example 4. Let $P(w)=sw+w^2,\ s>4$. Consider the equation

$$f(sz)=P(f(z))=sf(z)+f(z)^2,\qquad f(0)=0,\ f'(0)=1.$$

Let $\mathcal{O}^-(\{0\})=\bigcup_n P^{-n}(0)$ be the backward orbit of $\{0\}$ by $P(w)$. Write $P^{-1}(0)=\{b_1,b_2\}$, where $b_1=0, b_2=-s$. Write $P^{-1}(b_1)=\{b_{11},b_{12}\},\ P^{-1}(b_2)=\{b_{21},b_{22}\}$, where

$$b_{11} = 0, \quad b_{12} = -s - b_{11} = -s < 0;$$
$$b_{21} = \frac{-s + s\sqrt{1-4/s}}{2}, \ b_{22} = -s - b_{21}, \quad -s \le b_{21}, b_{22} \le 0.$$

Then $P^{-2}(0) = P^{-1}(b_1) \cup P^{-1}(b_2)$. In general, write $P^{-n}(0) = \{b_{J_n}; J_n \in \mathfrak{I}_n\}$, where $\mathfrak{I}_n$ denotes the set of n-tuples of $1, 2$. Suppose $-s \le b_{J_n} \le 0$. Then $P^{-n-1}(0) = \{b_{J_n 1}, b_{J_n 2}; J_n \in \mathfrak{I}_n\}$, in which

$$b_{J_n 1} = \frac{-s + s\sqrt{1 + 4b_{J_n}/s}}{2}, \ b_{J_n 2} = -s - b_{J_n 1},$$

hence $-s \le b_{J_{n+1}} \le 0$. Thus $\mathcal{O}^-(\{0\}) \subset [-s, 0]$. Let $f(z_0) = -s = b_2$. Then $f(z_0/s^m) = b_{J_n} \in \mathcal{O}^-(\{0\})$ for some b_{J_n}. There is $\delta > 0$ such that $f(z)$ is injective on $\{z; |z| < \delta\}$. Take m so large that $|z_0/s^m| < \delta$. Obviously $w = f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, where $a_k \in \mathbb{R}$. Hence $z = w + \sum_{k=2}^{\infty} c_k w^k$, where $c_k \in \mathbb{R}$, and

$$\frac{z_0}{s^m} = b_{J_n} + O(b_{J_n}^2) \le 0,$$

which shows that $z_0 \le 0$. Since $f(z)$ takes every $a \in [-s, 0]$ in $[z_0, 0]$, we get $\mathcal{O}^-(\{0\}) \subset f([z_0, 0])$. Therefore $\mathcal{J}_P = \overline{\mathcal{O}^-(\{0\})} \subset \overline{f([z_0, 0])} = f([z_0, 0]) \subset f(d_\pi)$, where $d_\pi = \{z; \arg[z] = \pi\}$. By the above arguments we see that every value in $\mathcal{O}^-(\{0\})$ is not taken by $f(z)$ other than in d_π, hence d_π is the only Julia direction for $f(z)$. Since the order $\rho(f) = \log 2/\log s$ of $f(z)$ is less than $1/2$, we see easily, by Wiman's theorem, e.g. [1, p. 39, Theorem 3.1.5], that $\mathcal{J}_P \subsetneq f(d_\pi)$. Note that $-s/2 \notin \mathcal{J}_P \subsetneq [-s, 0]$. In fact, $P(-s/2) = -s^2/2 + s^2/4 = -s^2/4 < -s$.

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