GROWTH OF MEROMORPHIC SOLUTIONS OF SOME FUNCTIONAL EQUATIONS I

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Summary. It is shown that each transcendental meromorphic solution $f(z)$ of the functional equation $\sum_{j=0}^{n} a_j(z)f(c^j z) = Q(z)$, where Q and the a_j, $j = 0, \ldots, n$ are polynomials without common zeros, $a_n(z)a_0(z) \neq 0$ and $0 < |c| < 1$, satisfies $m(r, f) = \sigma_f(\log r)^2(1 + o(1))$ for some constant σ_f.

1. Introduction

In this paper we are concerned with meromorphic solutions of the functional equation

$$\sum_{j=0}^{n} a_j(z)f(c^j z) = Q(z), \tag{1.1}$$

where Q and the a_j, $j = 0, \ldots, n$ are polynomials without common zeros, $a_n(z)a_0(z) \neq 0$ and $0 < |c| < 1$. Throughout this paper, we use standard notations in the Nevanlinna theory (see, e.g., [3], [7], [8], [9]). Let $f(z)$ be a meromorphic function. Let $m(r, f)$, $n(r, f)$, $N(r, f)$ and $T(r, f)$ denote the proximity function, the unintegrated counting function, the counting function and the characteristic function of $f(z)$ respectively, and $M(r, f)$ denote the maximum modulus of $f(z)$, when $f(z)$ is entire. For two real functions $\phi(r)$ and $\psi(r)$, $r \in \mathbb{R}^+$, we write $\phi(r) \sim \psi(r)$ if $\phi(r)/\psi(r) \to 1$ as $r \to \infty$. In the paper [1], we proved

Theorem A. *Any meromorphic solution $f(z)$ of (1.1) satisfies $T(r, f) = O((\log r)^2)$.*

This theorem is generalized in [6] to the case when $a_j(z)$ are transcendental functions. We also proved the following theorem which gives a "lower bound" of the characteristic functions of transcendental solutions.

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Theorem B. *Any transcendental meromorphic solution $f(z)$ of (1.1) satisfies $(\log r)^2 = O(T(r, f))$.*

For the case $n = 1$ in (1.1), Wittich [11] treated entire solutions of the functional equation

$$f(sz) = P_1(z)f(z) + P_0(z), \tag{1.2}$$

where $P_1(z)$ and $P_0(z)$ are polynomials, and $|s| > 1$. Wittich proved that all solutions $f(z)$ of (1.2) satisfy $\log M(r, f) \sim \dfrac{\tau}{2\log|s|}(\log r)^2$, as $r \to \infty$, where τ is the degree of P_1.

It would be natural to ask, for a solution $f(z)$ of (1.1), whether there exists a constant σ_f such that $T(r, f) \sim \sigma_f (\log r)^2$ or not. We will give answers to this question in this note.

At first we have

Theorem 1.1. *Suppose* (1.1) *possesses a transcendental entire solution $f(z)$. Then there is some j, $1 \le j \le n$, such that* $\deg[a_0(z)] < \deg[a_j(z)]$.

Put $p_j = \deg[a_j]$ and $d_j = p_j - p_0$. By Theorem 1.1 there exists j such that $d_j > 0$ if (1.1) admits a transcendental entire solution.

For $j \in \{0, \dots, n\}$, let

$$P_j = \{(x, y) \mid x \ge j \text{ and } y \le d_j\},$$

and define P to be the convex hull of $\bigcup_{j=0}^{n} P_j$. P is called the *Newton-Puiseux diagram.*

Let (j_k, d_{j_k}), $k = 0, \dots, m$, be the vertices of P, where $0 = j_0 < j_1 < \cdots < j_m \le n$. To simplify notation, we use the abbreviation $D_k = d_{j_k}$ for $k = 0, \dots, m$. For $k = 1, \dots, m$ we define

$$\sigma_k = \frac{D_k - D_{k-1}}{j_k - j_{k-1}}.$$

Then $\sigma_1 > \sigma_2 > \cdots > \sigma_m > 0$. The σ_k are the slopes of the segments which form the boundary of P (this boundary is called the *Newton-Puiseux polygon*). It follows from the definition of the σ_k that

$$d_j \le D_k + \sigma_k(j - j_k) = D_{k-1} + \sigma_k(j - j_{k-1})$$

for all $j \in \{0, 1, \dots, n\}$ and all $k \in \{1, \dots, m\}$, with strict inequality if $j < j_{k-1}$ or $j > j_k$. Moreover, we have $d_j \le D_m$ for all j. Finally, for $k = 1, \dots, m$ we define

$$\tau_k = \frac{\sigma_k}{-2\log|c|}.$$

It will also be convenient to define $\sigma_{m+1} = \tau_{m+1} = 0$.

Theorem 1.2. *Suppose that* (1.1) *possesses a transcendental entire solution* f*. Then there exists* $k \in \{1, \ldots, m\}$ *such that*

$$\log M(r, f) \sim \tau_k (\log r)^2. \tag{1.3}$$

Ramis [10] treated the functional equation (1.1) by means of methods from functional analysis, and obtained $\limsup_{r\to\infty} \dfrac{\log M(r, f)}{(\log r)^2} = \tau_k$.

Theorem 1.3. *Suppose that* (1.1) *possesses a transcendental meromorphic solution* f*. Then there exists a constant* σ *such that*

$$m(r, f) \sim \sigma(\log r)^2. \tag{1.4}$$

A corresponding result for $N(r, f)$ is proved in [2]. Combining these two results one concludes that a transcendental meromorphic solution f of (1.1) satisfies $T(r, f) \sim c(\log r)^2$ for some $c > 0$ as r tends to infinity.

We prove Theorem 1.1 in Section 2, Theorem 1.2 in Section 3, and Theorem 1.3 in Section 4.

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2. Proof of Theorem 1.1

Let

$$Q(z) = \sum_{k=0}^{q} Q_k z^k \tag{2.1}$$

and

$$a_j(z) = b_{j0} - \sum_{k=1}^{p_j} b_{jk} z^k.$$

Write the transcendental entire solution $f(z)$, which is supposed to exist, as

$$f(z) = \sum_{\nu=0}^{\infty} \alpha_\nu z^\nu. \tag{2.2}$$

Put

$$T_\nu = b_{00} + b_{10}c^\nu + b_{20}c^{2\nu} + \cdots + b_{n0}c^{n\nu}. \tag{2.3}$$

Then we have

$$\begin{aligned} T_0\alpha_0 &= Q_0, \\ T_1\alpha_1 &= (b_{01}+b_{11}+\cdots+b_{n1})\alpha_0 + Q_1, \\ T_\nu\alpha_\nu &= \sum_{j=0}^{n}\left(\sum_{k=1}^{\min(\nu,p_j)} b_{jk}c^{(\nu-k)j}\alpha_{\nu-k}\right) + Q_\nu \quad (0\le\nu\le q) \end{aligned}$$

and, for sufficiently large ν,

$$T_\nu\alpha_\nu = \sum_{j=0}^{n}\left(\sum_{k=1}^{p_j} b_{jk}c^{(\nu-k)j}\alpha_{\nu-k}\right). \tag{2.4}$$

Now suppose, on the contrary to the assertion, that $p_0 \ge p_j$ $(1\le j\le n)$. Since (2.2) is entire, we have

$$\limsup_{\nu\to\infty}|\alpha_\nu|^{1/\nu} = 0. \tag{2.5}$$

Put

$$\eta_\nu = |\alpha_\nu|^{1/\nu} \quad\text{and}\quad \xi_\nu = \sup_{\mu\ge\nu}|\alpha_\mu|^{1/\mu}.$$

By (2.5), $\eta_\nu \le \xi_\nu \downarrow 0$. From (2.4) we have (putting $b_{jk}=0$ for $k>p_j$),

$$\left|b_{0p_0}+\sum_{j=1}^{n}b_{j,p_0}c^{j(\nu-p_0)}\right|\eta_{\nu-p_0}^{\nu-p_0} \le |T_\nu|\xi_\nu^\nu + \sum_{k=1}^{p_0-1}\left|\sum_{j=1}^{n}b_{jk}c^{j(\nu-k)}\right|\xi_{\nu-k}^{\nu-k}. \tag{2.6}$$

Let

$$\mu(\nu) = \min\{\mu \ \mid \mu\ge\nu \text{ and } \eta_\mu = \xi_\nu\}.$$

Then $\eta_{\mu(\nu)} = \xi_{\mu(\nu)}$. By (2.6), writing $\mu(\nu)$ simply as μ,

$$\left|b_{0p_0}+\sum_{j=1}^{n}b_{j,p_0}c^{j\mu}\right|(\xi_\mu)^\mu \le |T_{\mu+p_0}|\xi_{\mu+p_0}^{\mu+p_0} + \sum_{k=1}^{p_0-1}\left|\sum_{j=1}^{n}b_{jk}c^{j(\mu+p_0-k)}\right|\xi_{\mu+p_0-k}^{\mu+p_0-k},$$

that is,

$$\begin{aligned} &\left|b_{0p_0}+\sum_{j=1}^{n}b_{j,p_0}c^{j\mu}\right| \\ &\le |T_{\mu+p_0}|\xi_{\mu+p_0}^{p_0}\left(\frac{\xi_{\mu+p_0}}{\xi_\mu}\right)^\mu + \sum_{k=1}^{p_0-1}\left|\sum_{j=1}^{n}b_{jk}c^{j(\mu+p_0-k)}\right|\xi_{\mu+p_0-k}^{p_0-k}\left(\frac{\xi_{\mu+p_0-k}}{\xi_\mu}\right)^\mu \\ &\le |T_{\mu+p_0}|(\xi_{\mu+p_0})^{p_0} + \sum_{k=1}^{p_0-1}\left|\sum_{j=1}^{n}b_{jk}c^{j(\mu+p_0-k)}\right|(\xi_{\mu+p_0-k})^{p_0-k}, \end{aligned}$$

Letting $\mu(\nu)\to\infty$ $(\nu\to\infty)$, we get $|b_{0p_0}|\le 0$, a contradiction.

3. Proof of Theorem 1.2

The proof requires the following

Lemma 3.1. *Let $r_0 \geq 0$ and $S : [r_0, \infty) \to [1, \infty)$ be continuous and non-decreasing. Define $\sigma = \liminf_{r\to\infty} \dfrac{\log S(r)}{(\log r)^2}$ and $\tau = \limsup_{r\to\infty} \dfrac{\log S(r)}{(\log r)^2}$. Suppose that $0 < \tau < \infty$ and that $\alpha > 0$ satisfies $\sigma \leq \alpha \leq \tau$. Define $\beta = \sqrt{\alpha\tau}$. Then there exist for any $K > 1$ and any $\varepsilon > 0$ arbitrarily large r such that*

$$S(tr) \leq r^{2\beta \log t + \varepsilon} S(r), \quad \textit{for } 1 \leq t \leq K$$
$$S(tr) \leq r^{2\alpha \log t + \varepsilon} S(r), \quad \textit{for } \frac{1}{K} \leq t \leq 1.$$

Proof of Lemma 3.1 The idea is to use Pólya peaks for the function $h(r) = \exp\sqrt{\log S(r)}$, see e.g. [4]. This function has order $\sqrt{\tau}$ and lower order $\sqrt{\sigma}$. and the basic result on Pólya peaks thus implies that there exist Pólya peaks of order $\sqrt{\alpha}$ for $h(r)$. Moreover, proofs of the existence of Pólya peaks show that they can be chosen such that if $\delta > 0$, then $h(r) \geq r^{\sqrt{\alpha}-\delta}$ on the peaks. We thus find arbitrarily large r satisfying the last inequality such that $h(tr) \leq (1+\delta)t^{\sqrt{\alpha}}h(r)$ for $\frac{1}{K} \leq t \leq K$. Taking logarithms, using $\log(1+\delta) < \delta$ and squaring yields

$$\log S(tr) \leq \log S(r) + 2\sqrt{\log S(r)}(\sqrt{\alpha}\log t + \delta) + (\sqrt{\alpha}\log t + \delta)^2$$

for $\frac{1}{K} \leq t \leq K$. For large r we have $\sqrt{\log S(r)} \leq (\sqrt{\tau} + \delta)\log r$ and hence find

$$S(tr) \leq S(r) r^{2(\sqrt{\tau}+\delta)(\sqrt{\alpha}\log t + \delta)} e^{(\sqrt{\alpha}\log t+\delta)^2}$$

for $1 \leq t \leq K$. Choosing δ sufficiently small and r large we obtain the desired estimate. For $\frac{1}{K} \leq t \leq 1$ we also use the estimate $\sqrt{\log S(r)} \geq (\sqrt{\alpha} - \delta)\log r$ which follows immediately from $h(r) \geq r^{\sqrt{\alpha}-\delta}$. We obtain

$$\log S(tr) \leq \log S(r) + 2\sqrt{\alpha}\log t(\sqrt{\alpha} - \delta)\log r + 2\delta(\sqrt{\tau} + \delta)\log r + (\sqrt{\alpha}\log t + \delta)^2$$

and thus

$$S(tr) \leq S(r) r^{2\sqrt{\alpha}(\sqrt{\alpha}-\delta)\log t + 2\delta(\sqrt{\tau}+\delta)} e^{(\sqrt{\alpha}\log t+\delta)^2}$$

for $\frac{1}{K} \leq t \leq 1$. Again the desired conclusion follows for sufficiently small δ and large r. □

Remark. If we choose $\alpha = \tau$ in the lemma, then we have $\beta = \tau$.

Proof of Theorem 1.2 Let

$$\sigma = \liminf_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \quad \text{and} \quad \tau = \limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2}.$$

We know already [1] that $0 < \tau < \infty$. We assume that the conclusion of the theorem does not hold. Then either $\sigma = \tau \neq \tau_k$ for all k or $\sigma < \tau$. In the first case we choose $\alpha = \beta = \tau$. In the second case we choose $\alpha < \tau$ such that the interval $[\alpha, \tau)$ does not contain any τ_k and define $\beta = \sqrt{\alpha\tau}$. Let now $\ell = \min\{k \in \{0, \dots, m\} : \tau_{k+1} < \alpha\}$. If $\ell \neq 0$, then $\tau_\ell > \beta \geq \alpha > \tau_{\ell+1}$. We write the functional equation in the form

$$f(c^{j_\ell} z) = -\sum_{j=0}^{j_\ell - 1} \frac{a_j(z)}{a_{j_\ell}(z)} f(c^j z) - \sum_{j=j_{\ell+1}}^{n} \frac{a_j(z)}{a_{j_\ell}(z)} f(c^j z) - \frac{Q(z)}{a_{j_\ell}(z)}.$$

Here the second sum is empty if $\ell = m$ and $j_m = n$.

Taking the maximum modulus yields for large r that

$$M(r,f) \leq C\left(\sum_{j=0}^{j_\ell-1} r^{d_j - D_\ell} M(|c|^{j-j_\ell} r, f) + \sum_{j=j_\ell+1}^{n} r^{d_j - D_\ell} M(|c|^{j-j_\ell} r, f) + r^{q-D_\ell}\right)$$

where $C > 0$ and $q \in \mathbb{Z}$. Because f is transcendental, we may omit the term r^{q-D_ℓ} by replacing C by a larger constant, if necessary.

We now apply Lemma 3.1 to $S(r) = M(r,f)$ and choose r as there. We obtain

$$M(r,f) \leq CM(r,f)\left(\sum_{j=0}^{j_\ell-1} r^{d_j - D_\ell + 2\beta(j-j_\ell)\log|c| + \varepsilon} + \sum_{j=j_\ell+1}^{n} r^{d_j - D_\ell + 2\alpha(j-j_\ell)\log|c| + \varepsilon}\right).$$

For all j we have

$$d_j - D_\ell \leq \sigma_\ell(j - j_\ell) = -2\log|c|\tau_\ell(j - j_\ell)$$

and

$$d_j - D_\ell \leq \sigma_{\ell+1}(j - j_\ell) = -2\log|c|\tau_{\ell+1}(\ell - j_\ell).$$

We use the first inequality for $j \leq j_\ell - 1$ and the second one for $j \geq j_\ell + 1$ and obtain

$$1 \leq C\left(\sum_{j=0}^{j_\ell-1} r^{-2\log|c|(j-j_\ell)(\tau_\ell - \beta) + \varepsilon} + \sum_{j=j_\ell+1}^{n} r^{-2\log|c|(j-j_\ell)(\tau_{\ell+1} - \alpha) + \varepsilon}\right).$$

For sufficiently small ε the exponents on the right hand side are all negative and thus we obtain a contradiction for large r.

If $\ell = 0$, we can make essentially the same argument. In this case, we have $j_0 = 0$ and the first sum in the above estimates is empty. □

Remark. With the above method one could also give an alternative proof of Theorem 1.1. In fact, if $p_j \leq p_0$ for all j, then $P = \{(x,y) \ \mid x \geq 0 \text{ and } y \leq 0\}$ is degenerate in the sense that $m = 0$, but suitable modifications of the case $\ell = 0$ above will work. Instead, we have preferred to give a direct proof of Theorem 1.1.

4. Proof of Theorem 1.3

We write the equation (1.1) in the form

$$a_n(z)f(c^n z) + \cdots + a_0(z)f(z) = Q(z),$$

in which $a_j(z)$ and $Q(z)$ are polynomials, without common zeros. We can assume $a_0(z) = z^\mu a_0^*(z),\ \mu \geq 0, a_0^*(0) = 1$, without losing generality. Putting

$$h(z) = \prod_{k=0}^{\infty} a_0^*(c^k z), \quad g(z) = h(z)f(z),$$

we obtain

$$A_n(z)g(c^n z) + \cdots + A_1(z)g(cz) + g(z) = Q(z)h_1(z), \tag{4.1}$$

where

$$A_1(z) = \frac{a_1(z)}{z^\mu}, \quad A_j(z) = \frac{a_j(z)}{z^\mu}\prod_{k=1}^{j-1} a_0^*(c^k z), \quad h_1(z) = z^{-\mu}\prod_{k=1}^{\infty} a_0^*(c^k z).$$

To begin with, we consider the homogeneous case

$$g(z) + A_1(z)g(cz) + \cdots + A_n(z)g(c^n z) = 0. \tag{4.1$'$}$$

Obviously the coefficients of (4.1′) are polynomials divided by z^μ:

$$\deg[A_j] = \deg[a_j] + (j-1)\deg[a_0^*] - \mu = d_j + j\deg[a_0^*], \quad 1 \leq j \leq n,$$

where $d_j = \deg[a_j] - \deg[a_0] = \deg[a_j] - \deg[a_0^*] - \mu$. A solution $g(z)$ of (4.1′) does not admit any pole for $z \neq 0$. In fact, suppose $z_0 \neq 0$ would be a pole, with the smallest modulus. Then one of $g(c^j z),\ 1 \leq j \leq n$, has a pole at $z = z_0$, whence $g(z)$ would have a smaller pole $c^j z_0$, a contradiction.

When $g(z)$ has a pole at $z = 0$ of order p, put $G(z) = z^p g(z)$. Then we get

$$G(z) + c^{-p}A_1(z)G(cz) + \cdots + c^{-pn}A_n(z)G(c^n z) = 0$$

with entire solution $G(z)$. Hence we suppose (4.1′) admits an entire solution $g(z)$.

Write $\deg[a_0^*] = q$ and

$$a_0^*(z) = \prod_{\ell=1}^{m}(1 - b_\ell z)^{q_\ell}, \quad q_1 + \cdots + q_m = q,$$

where $1/b_\ell$ are zeros of $a_0(z)$ with multiplicity q_ℓ. Then

$$h(z) = \prod_{\ell=1}^{m}\left(\prod_{k=0}^{\infty}(1 - c^k b_\ell z)^{q_\ell}\right).$$

By Theorem 1.2, we have that

$$\log M(r,g) \sim \tau_g(\log r)^2, \quad \tau_g = \sigma^*/(2\log s), \quad s = 1/|c|, \tag{4.2}$$

where σ^* is one of the slopes of the Newton polygon with respect to (4.1′). Let $(j_1, d_{j_1} + j_1 q)$ and $(j_2, d_{j_2} + j_2 q)$ be vertices of the Newton polygon to (4.1′) which give σ^*. We pay attention to the corresponding slope σ of the Newton polygon to (1.1). We obtain

$$\sigma^* = \frac{(d_{j_2} + j_2 q) - (d_{j_1} + j_1 q)}{j_2 - j_1} = \sigma + q$$

Since $h(z)$ satisfies the functional equation $-a_0^*(z)h(cz) + h(z) = 0$, we have

$$\log M(r,h) \sim \tau_h (\log r)^2, \quad \tau_h = q/(2\log s). \tag{4.3}$$

We assert that

$$m(r,f) \sim \frac{\max(\sigma^* - q, 0)}{2\log s}(\log r)^2 = \max(\tau_g - \tau_h, 0)(\log r)^2, \tag{4.4}$$

which gives the assertion of Theorem 1.3 for the homogeneous case.

To prove (4.4), we shall show that (i) $m(r,f) \le \max(\tau_g - \tau_h, 0)(\log r)^2(1+o(1))$, and (ii) $m(r,f) \ge \max(\tau_g - \tau_h, 0)(\log r)^2(1+o(1))$. In order to prove (i), we need Lemma 4.1 below.

Lemma 4.1. *Let $N > 0$ be a fixed integer and let $\{r_j\}$ be a sequence satisfying $r_j < r_{j+1}$, $j = 0, 1, 2, \ldots$. We define for $k \in \mathbb{N}$*

$$G_k(z) = \prod_{j=0}^{N} \left(1 + \frac{z}{r_{k+j}}\right).$$

Then we have for $r_k \le r < r_{k+N}$

$$m\left(r, \frac{1}{G_k}\right) \le C_1 \log\left(\frac{r_{k+N}}{r_k}\right) + C_2,$$

where C_1, C_2 are constants independent of k and r.

Proof of Lemma 4.1. Since $|1 + (r/r_{k+j})e^{i\theta}|^2 \ge 4(r/r_{k+j})\cos^2(\theta/2)$, we have

$$\int_0^{2\pi} \log^+ \frac{1}{|G_k(re^{i\theta})|} d\theta \le \sum_{j=0}^{N} \int_0^{2\pi} \log^+ \left| \frac{1}{\sqrt{\frac{r}{r_{k+j}}}\cos\frac{\theta}{2}} \right| d\theta.$$

We note that $\int_0^{2\pi} \log|1/\cos(\theta/2)|d\theta$ is a finite constant, and we name it C_0. In the case $|r/r_{k+j}| \le 1$, and hence $|\sqrt{\frac{r}{r_{k+j}}}\cos\frac{\theta}{2}| \le 1$, we have

$$\begin{aligned}
\int_0^{2\pi} \log^+ \left| \frac{1}{\sqrt{\frac{r}{r_{k+j}}}\cos\frac{\theta}{2}} \right| d\theta &= \int_0^{2\pi} \log \left| \frac{1}{\sqrt{\frac{r}{r_{k+j}}}\cos\frac{\theta}{2}} \right| d\theta = \int_0^{2\pi} \log\sqrt{\frac{r_{k+j}}{r}} d\theta + C_0 \\
&= \pi \log\left(\frac{r_{k+j}}{r}\right) + C_0 \le \pi \log\left(\frac{r_{k+N}}{r_k}\right) + C_0.
\end{aligned}$$

If $|r/r_{k+j}| > 1$, then we have

$$\int_0^{2\pi} \log^+ \left| \frac{1}{\sqrt{\frac{r}{r_{k+j}}} \cos\frac{\theta}{2}} \right| d\theta \leq C_0.$$

Hence we obtain that

$$m\left(r, \frac{1}{G_k}\right) \leq \frac{N}{2} \log\left(\frac{r_{k+N}}{r_k}\right) + \frac{N}{2\pi} C_0,$$

and the assertion of Lemma 4.1 follows. □

(i) Let $p \in \mathbb{N}$ be such that $s^{-p} \leq |b_\ell| < s^p, \ell = 1, \dots, m$. For any z, we find k satisfying $s^k \leq |z| < s^{k+1}$. We define

$$\Pi_1^{(k)}(z) = \prod_{\ell=1}^{m} \left(\prod_{j=0}^{k-p-1} (1 - b_\ell c^j z)^{q_\ell} \right) \quad \text{and} \quad \Pi_2^{(k)}(z) = \prod_{\ell=1}^{m} \left(\prod_{j=k+p+2}^{\infty} (1 - b_\ell c^j z)^{q_\ell} \right),$$

and we write $h(z) = \Pi_1^{(k)}(z) F_k(z) \Pi_2^{(k)}(z)$, where

$$F_k(z) = \prod_{\ell=1}^{m} \left(\prod_{j=k-p}^{k+p+1} (1 - b_\ell c^j z)^{q_\ell} \right).$$

We show that $|\Pi_2^{(k)}|$ is bounded from above and below. For the sake of simplicity, we put $K_1 = \prod_{j=1}^{\infty} (1 - s^{-j})^q$, and $K_2 = \prod_{j=1}^{\infty} (1 + s^{-j})^q$. We have for $j \geq k+p+2$,

$$|1 - b_\ell c^j z| \leq 1 + |b_\ell| s^{-j} |z| < 1 + s^{k+p+1-j}, \quad \text{and}$$
$$|1 - b_\ell c^j z| \geq 1 - |b_\ell| s^{-j} |z| > 1 - s^{k+p+1-j}.$$

Using this we get

$$|\Pi_2^{(k)}(z)| \leq \prod_{\ell=1}^{m} \prod_{j=k+p+2}^{\infty} (1 + s^{k+p+1-j})^{q_\ell} = \left(\prod_{j=1}^{\infty} (1 + s^{-j}) \right)^{q_1 + \cdots + q_m} = K_2, \tag{4.5}$$

$$|\Pi_2^{(k)}(z)| \geq \prod_{\ell=1}^{m} \prod_{j=k+p+2}^{\infty} (1 - s^{k+p+1-j})^{q_\ell} = \left(\prod_{j=1}^{\infty} (1 - s^{-j}) \right)^{q_1 + \cdots + q_m} = K_1. \tag{4.6}$$

Next we try to find the lower bound of $|\Pi_1^{(k)}(z)|$. For the case $j \leq k - p - 1$, we have $|1 - b_\ell c^j z| \geq s^{k-p-j} - 1$. Hence we get

$$\begin{aligned} |\Pi_1^{(k)}(z)| &\geq \prod_{\ell=1}^{m} \prod_{j=0}^{k-p-1} (s^{k-p-j} - 1)^{q_\ell} = \left(\prod_{j=0}^{k-p-1} (s^{k-p-j} - 1) \right)^{q_1 + \cdots + q_m} \\ &= \prod_{j=1}^{k-p} (s^j - 1)^q = \prod_{j=1}^{k-p} s^{jq} (1 - s^{-j})^q \geq \prod_{j=1}^{k-p} s^{jq} K_1 = K_1 s^{q(1 + \cdots + (k-p))} \\ &= K_1 s^{q(k-p)(k-p+1)/2}. \end{aligned}$$

Therefore we obtain

$$(4.7)\qquad |\Pi_1^{(k)}(z)| \geq s^{(q/2)(\log r/\log s)^2(1+o(1))} = e^{\tau_h(\log r)^2(1+o(1))},$$

where $\tau_h = q/(2\log s)$. Now we estimate the proximity function of $f = g/h$ by Lemma 4.1, (4.5), (4.6) and (4.7):

$$\begin{aligned} m(r,f) = m(r,\frac{g}{h}) &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{g(re^{i\theta})}{\Pi_1^{(k)}(re^{i\theta})F_k(re^{i\theta})\Pi_2^{(k)}(re^{i\theta})}\right|d\theta + O(\log r) \\ &\leq \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{g(re^{i\theta})}{\Pi_1^{(k)}(re^{i\theta})}\right|d\theta + \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{1}{F_k(re^{i\theta})}\right|d\theta \\ &+ \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{1}{\Pi_2^{(k)}(re^{i\theta})}\right|d\theta + O(\log r) \\ &\leq \frac{1}{2\pi}\int_0^{2\pi}\log^+\left(\frac{M(r,g)}{|\Pi_1^{(k)}(re^{i\theta})|}\right)d\theta + O(\log r) \\ &\leq \frac{1}{2\pi}\int_0^{2\pi}\log^+|e^{\max(\tau_g-\tau_h,0)(\log r)^2(1+o(1))}|d\theta + O(\log r) \\ &= \max(\tau_g-\tau_h,0)(\log r)^2(1+o(1)). \end{aligned}$$

(ii) For any r, we have $m(r,g) \leq m(r,f) + m(r,h) + O(\log r)$, i.e.

$$m(r,f) \geq m(r,g) - m(r,h) + O(\log r).$$

We fix $\delta > 1$ arbitrary and set $\rho = r(\delta+1)/(\delta-1) > r$, $(\delta = (\rho+r)/(\rho-r))$. Then

$$(4.8)\qquad \begin{aligned} m(\rho,f) &\geq m(\rho,g) - m(\rho,h) + O(\log\rho) \\ &\geq \frac{\rho-r}{\rho+r}\log M(r,g) - m(\rho,h) + O(\log\rho) \\ &\geq \frac{1}{\delta}\log M(r,g) - \log M(\rho,h) + O(\log\rho). \end{aligned}$$

Using (4.2), (4.3) and (4.8), we have

$$\begin{aligned} m(\rho,f) &\geq \frac{\tau_g}{\delta}(\log r)^2(1+o(1)) - \tau_h(\log\rho)^2(1+o(1)) + O(\log\rho) \\ &\geq \left(\frac{\tau_g}{\delta}\left(\frac{\log r}{\log\rho}\right)^2 - \tau_h\right)(\log\rho)^2(1+o(1)) \end{aligned}$$

Hence, letting $\rho \to \infty$, we obtain

$$\liminf_{\rho\to\infty}\frac{m(\rho,f)}{(\log\rho)^2} \geq \frac{\tau_g}{\delta} - \tau_h.$$

Since $\delta > 1$ is arbitrary, we have

$$\liminf_{\rho\to\infty}\frac{m(\rho,f)}{(\log\rho)^2} \geq \tau_g - \tau_h.$$

Therefore we obtain

$$m(r,f) \geq (\tau_g - \tau_h)(1+o(1))(\log r)^2.$$

Thus we obtain (4.4).

Next we treat the inhomogeneous case $Q(z) \not\equiv 0$ in (1.1) and (4.1). Set $z \to cz$ in (4.1), i.e.

$$\sum_{j=0}^{n} A_j(cz)g(c^{j+1}z) = h_1(cz)Q(cz), \quad A_0(z) \equiv 1. \tag{4.9}$$

It follows from (4.9) and the functional equation of $h(z)$ that

$$\sum_{j=0}^{n} A_j(cz)g(c^{j+1}z) = h_1(z)Q(cz)/H(cz),$$

where $H(z) = c^{\mu}a_0^*(z)$. Combining (4.1) and the equation above, we get the following homogeneous equation

$$H(cz)Q(z)A_n(cz)g(c^{n+1}z) + \sum_{j=1}^{n} B_j(z)g(c^j z) - Q(cz)A_0(z)g(z) = 0, \tag{4.10}$$

where $B_j(z) = A_{j-1}(cz)H(cz)Q(z) - Q(cz)A_j(z)$. In view of Theorem 1.2, we get

$$\log M(r,g) \sim \tilde{\tau}_g(\log r)^2, \quad \tilde{\tau}_g = \tilde{\sigma}^*/(2\log s),$$

where $\tilde{\sigma}^*$ a slope of the Newton polygon for (4.10). By means of the result in the homogeneous case, we obtain

$$m(r,f) \sim (\tilde{\tau}_g - \tau_h)(\log r)^2.$$

Remark. Let $L(r,f) = \min_{|z|=r}|f(z)|$ be the minimum modulus of an entire funtion f. If f has order 0, and thus in particular if (1.3) is satisfied, we have $\log L(r,f) \sim \log M(r,f)$ as $r \to \infty$ outside some exceptional set of logarithmic density zero; see [5] for a detailed account of such minimum modulus theorems. It thus follows from (4.2) and (4.3) that (4.4) holds outside an exceptional set of logarithmic density zero. The method used above is more elementary in that it does not use such minimum modulus theorems, and it also avoids the occurence of exceptional sets.

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