

Deficiency for Meromorphic Solutions of Schröder Equations*

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Eremenko and Sodin proved that meromorphic solution $f(z)$ of the Schröder equation $f(sz) = R(f(z))$, $|s| > 1$, has no Valiron deficiency other than exceptional values of $R(z)$. We consider transcendental meromorphic solutions of non-autonomous equation $f(sz) = R(z, f(z))$, $|s| > 1$. It is shown that there exists an equation of this form possessing a transcendental meromorphic solution, which has a Valiron deficiency other than a Nevanlinna deficiency. We also give some generalizations of the Eremenko and Sodin theorem for algebraic functions as targets.

Keywords: Schröder equation; Meromorphic functions; Nevanlinna theory; Valiron deficiency; Algebraic functions

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1. INTRODUCTION

Let s be a fixed complex number with $|s| > 1$. In this note we consider a functional equation

$$f(sz) = R(z, f(z)), \quad (1.1)$$

where $R(z, w)$ is a rational function in z and w with $\deg_w[R] = d \geq 2$. We assume that $R(z, w)$ is holomorphic at $(0, 0)$

$$R(z, w) = \sum_{j,k=0}^{\infty} \alpha_{j,k} z^j w^k, \quad |z| < \delta, |w| < \eta.$$

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The Eq. (1.1) admits a solution supposed that $w = R(0, w)$ has a finite root γ_0 and further that $s^n - \alpha_{0,1} \neq 0$ for any $n \geq 1$ if $\alpha_{1,0} \neq 0$, and $s - \alpha_{0,1} = 0$, $s^n - \alpha_{0,1} \neq 0$ for any $n \geq 2$ if $\alpha_{1,0} = 0$, [7, p.152]. In the autonomous case, that is to say, $R(z, w)$ does not contain z , (1.1) is the Schröder equation. We call (1.1) a generalized Schröder equation, in particular $R(z, w)$ contains z , we call a non-autonomous Schröder equation, see for example [2, 5]. For usual notations of Nevanlinna theory, $m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, $T(r, f)$, and for $\alpha \in \mathbb{C}$, $m(r, \alpha; f)$, $N(r, \alpha; f)$, $\overline{N}(r, \alpha; f)$ etc., we refer to for example [3, 4].

We assume that (1.1) has a transcendental meromorphic solution $f(z)$. Then we have that the growth order $\rho = \rho(f)$ is equal to $\log d / \log |s|$ and the Nevanlinna characteristic function $T(r, f)$ satisfies

$$K_1 r^\rho \leq T(r, f) \leq K_2 r^\rho, \quad (1.2)$$

with some constant K_1 and K_2 , see for example [7, p. 160].

We call $\alpha \in \mathbb{C} \cup \{\infty\}$ *Picard exceptional value* if $f(z) \neq \alpha$ for any $z \in \mathbb{C}$. It is said to be a *Nevanlinna deficiency*, and said to be a *Valiron deficiency*, if

$$\liminf_{r \rightarrow \infty} \frac{m(r, \alpha; f)}{T(r, f)} > 0 \quad \text{and} \quad \limsup_{r \rightarrow \infty} \frac{m(r, \alpha; f)}{T(r, f)} > 0,$$

respectively. For a meromorphic function $f(z)$, we denote by $E_P(f)$, $E_N(f)$ and $E_V(f)$ the set of Picard exceptional values, Nevanlinna deficiencies and Valiron deficiencies. The set $E(R)$ of exceptional values of rational function $R(z)$ consists of those $a \in \mathbb{C} \cup \{\infty\}$ such that the equation $R^{\circ n}(z) = a$, $n \in \mathbb{N}$ have totally a finite number of roots, where $R^{\circ n}(z)$ denotes the n th iteration of $R(z)$. Let $f(z)$ be a transcendental meromorphic solution of the Schröder equation $f(sz) = R(f(z))$, $|s| > 1$. We call $f(z)$ here a Schröder function for $R(z)$. We have that $E_P(f) = E(R)$. By definition we have $E_P(f) \subset E_N(f) \subset E_V(f)$. One of the authors, Yanagihara [9] proved that in the autonomous case the Schröder function has no Nevanlinna deficiency different from Picard exceptional values, which gives that $E_P(f) = E_N(f)$. Eremenko and Sodin [1] improved Yanagihara's result that the Schröder function has no Valiron deficiency other than exceptional values of $R(z)$. This implies that $E(R) = E_P(f) = E_N(f) = E_V(f)$. One purpose of this note is to show that $E_N(f) = E_V(f)$ does not always hold in the non-autonomous case.

PROPOSITION 1 *There exists a transcendental meromorphic solution of a non-autonomous Schröder equation satisfying $E_N(f) \subsetneq E_V(f)$.*

We prove Proposition 1 in Section 3. Next we obtain a generalization of Eremenko and Sodin's result for an algebraic function. Let $a(z)$ be an algebraic function defined by an irreducible polynomial in w with rational function coefficients

$$H(z, w) = w^p + \cdots + h_1(z)w + h_0(z) = 0, \quad h_k(z) \in \mathbb{C}(z), \quad 0 \leq k \leq p-1. \quad (1.3)$$

Then the proximity function $m(r, a; f)$ of $f(z)$ to $a(z)$ is defined by

$$m(r, a; f) = \frac{1}{p} m(r, 0; H(z, f(z))). \quad (1.4)$$

We call z_0 is a $a(z)$ -point of $f(z)$ if $H(z_0, f(z_0)) = 0$. Denoted by $n(r, a; f)$ the number of zeros of $H(z, f(z))$ in $|z| \leq r$, divided by p , and define the counting function $N(r, a; f)$ by

$$\begin{aligned} N(r, a; f) &= \int_0^r \frac{n(t, a; f) - n(0, a; f)}{t} dt + n(0, a; f) \log r \\ &= \frac{1}{p} \int_0^r \frac{n(t, 0; H(z, f(z))) - n(0, 0; H(z, f(z)))}{t} dt + n(0, 0; H(z, f(z))) \log r. \end{aligned} \quad (1.5)$$

By the Valiron–Mohon'ko theorem, we have $T(r, H(z, f(z))) = pT(r, f) + O(\log r)$, see for example [3, p. 29]. Then from (1.4) and (1.5),

$$T(r, f) = m(r, a; f) + N(r, a; f) + O(\log r).$$

An algebraic function $a(z)$, defined by (1.3), is said to be a *Picard exceptional function* for $f(z)$, if $H(z, f(z))$ has no zeros, i.e. $N(r, a; f) = 0$. Put

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{m(r, a; f)}{T(r, f)} &= 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; f)}{T(r, f)} = \delta(a; f), \\ \limsup_{r \rightarrow \infty} \frac{m(r, a; f)}{T(r, f)} &= 1 - \liminf_{r \rightarrow \infty} \frac{N(r, a; f)}{T(r, f)} = \Delta(a; f). \end{aligned}$$

When $\delta(a; f) > 0$, $a(z)$ is said to be a *Nevanlinna deficient function* for $f(z)$, and when $\Delta(a; f) > 0$, $a(z)$ is said to be a *Valiron deficient function* for $f(z)$. We denote by $E_p^*(f)$, $E_N^*(f)$ and $E_V^*(f)$ the set of Picard exceptional functions, Nevanlinna deficient functions and Valiron deficient functions. Further we call $a(z)$ a *totally ramified Picard function* if $\overline{N}(r, a; f) = o(T(r, f))$ as $r \rightarrow \infty$ including Picard functions and denote by $E_p^*(f)$ the set of totally ramified Picard functions. By definition we have $E_p^*(f) \subset E_p^*(f)$. Our second result is the following.

THEOREM 2 *A transcendental meromorphic solution $f(z)$ of (1.1) has no Valiron deficient function other than totally ramified Picard functions. That is to say,*

$$m(r, a; f) = o(T(r, f)), \quad r \rightarrow \infty, \quad (1.6)$$

if $a(z)$ is not a totally ramified Picard exceptional function.

Note that even though $a(z)$ is a totally ramified Picard function, it may be $N(r, a; f) \neq o(T(r, f))$ in general, see proof of Proposition 1. Theorem 2 implies that $E_p^*(f) \subset E_N^*(f) \subset E_V^*(f) \subset E_p^*(f)$.

2. PRELIMINARIES

Denote by \mathfrak{K} the field of algebraic functions. Let $a(z) \in \mathfrak{K}$ be defined by $H(z, w) = 0$ in (1.3), and let $R(z, w)$ be a rational function in z and w . We consider algebraic

functions defined by $H(z, R(z, w)) = 0$. We have

$$\begin{aligned} H(z, R(z, w)) &= \frac{K(z)\tilde{H}_1(z, w)^{k_1} \cdots \tilde{H}_q(z, w)^{k_q}}{Q(z, w)} \\ &= \frac{K(z)H_1(z/s, w)^{k_1} \cdots H_q(z/s, w)^{k_q}}{Q(z, w)}, \end{aligned} \quad (2.1)$$

where $K(z) \in \mathbb{C}(z)$, $H(z/s, w) = \tilde{H}_j(z, w)$, $j = 1, \dots, q$ are relatively prime irreducible polynomials in w of the form (1.3), and $Q(z, w)$ is a polynomial in z and w . We may assume $\tilde{Q}(z, w)$ is a polynomial in w with polynomial coefficients having no common zeros. If $\deg_w[H] = p$ and $\deg_w[H_j] = p_j$, then we have obviously $p_1k_1 + \cdots + p_qk_q \leq pd$. In particular, $p_jk_j \leq pd$.

Let $a_k(z)$, $k = 1, 2$, be different branches of $a(z)$. Since $H(z, R(z, c_k^{[j]}(z))) = 0$, there are branches $c_k^{[j]}(z)$, $k = 1, 2$, such that $a_k(z) - R(z, c_k^{[j]}(z)) = 0$. If $a_1(z_0) \neq a_2(z_0)$, then we must have $c_1^{[j]}(z_0) \neq c_2^{[j]}(z_0)$. Hence the number of branches of $c^{[j]}(z)$ is not smaller than that of $a(z)$, which implies $p \leq p_j$, $1 \leq j \leq q$. In particular, $k_1 + \cdots + k_q \leq d$. If $q \geq 2$, then $k_j \leq d - 1$.

For example, we consider the case $H(z, w) = w^2 - z^3$, $R(z, w) = w^2 + zw - z$. By simple computation, $H(z, R(z, w)) = \tilde{H}_1(z, w)\tilde{H}_2(z, w)$, where $\tilde{H}_1(z, w) = w^2 - z$ and $\tilde{H}_2(z, w) = w^2 + 2zw + z^2 - z$. We set $z_0 = 1$ and choose branches of algebraic function defined by $H(z, w) = 0$, $a_1(z) = z\sqrt{z}$ and $a_2(z) = -z\sqrt{z}$. Then we see that corresponding branches of algebraic functions defined by $\tilde{H}_1(z, w)$ and $\tilde{H}_2(z, w)$ are $c_1^{[1]}(z) = \sqrt{z}$, $c_2^{[1]}(z) = -\sqrt{z}$, $c_1^{[2]}(z) = -z - \sqrt{z}$ and $c_2^{[2]}(z) = -z + \sqrt{z}$.

Suppose that there is a $\alpha(z) \in \mathfrak{K}$ defined by $H_\alpha(z, w) = 0$ such that $\alpha(z) - R(z, w) = 0$ is only given by $w = \alpha(z/s)$. We call such an algebraic function α a *maximally fixed algebraic function for $R(z, w)$* . In other words, any algebraic function defined by $H_\alpha(z, R(z, w)) = 0$ must be $\alpha(z/s)$. Hence by (2.1), we have

$$H_\alpha(z, R(z, w)) = \frac{K_\alpha(z)H_\alpha(z/s, w)^d}{Q_\alpha(z, w)}, \quad K_\alpha(z) \in \mathbb{C}(z). \quad (2.2)$$

Suppose that there exists a pair of algebraic functions $c(z)$ defined by $H_c(z, w) = 0$ and $\mathfrak{d}(z)$ defined by $H_{\mathfrak{d}}(z, w) = 0$ such that $c(z) - R(z, w) = 0$ is only given by $w = \mathfrak{d}(z/s)$, and $\mathfrak{d}(z) - R(z, w) = 0$ is only given by $w = c(z/s)$. We call such algebraic functions c and \mathfrak{d} a *maximally fixed pair of algebraic functions for $R(z, w)$* . We have

$$H_c(z, R(z, w)) = \frac{K_c(z)H_{\mathfrak{d}}(z/s, w)^d}{Q_c(z, w)}, \quad K_c(z) \in \mathbb{C}(z), \quad (2.3)$$

$$H_{\mathfrak{d}}(z, R(z, w)) = \frac{K_{\mathfrak{d}}(z)H_c(z/s, w)^d}{Q_{\mathfrak{d}}(z, w)}, \quad K_{\mathfrak{d}}(z) \in \mathbb{C}(z). \quad (2.4)$$

Suppose that (1.1) has a transcendental meromorphic solution $f(z)$. We consider $a(z/s)$ -points of $f(z)$, namely zeros of $H(z, f(sz))$. From (1.1) and (2.1),

$$\begin{aligned} H(z, f(sz)) &= H(z, R(z, f(z))) \\ &= \frac{K(z)H_1(z/s, f(z))^{k_1} \cdots H_q(z/s, f(z))^{k_q}}{Q(z, f(z))}, \end{aligned} \quad (2.5)$$

where $K(z) \in \mathbb{C}(z)$, $H_j(z, w)$, $j = 1, \dots, q$ are relatively prime polynomials in w of the form (1.3), and $Q(z, w)$ is a polynomial in z and w . Then $H_j(z, w) = 0$ defines $c^{[j]}(z) \in \mathfrak{K}$. We see that $a(z)$ -points of $f(sz)$ are from $c^{[j]}(z/s)$ -points of $f(z)$ with multiplicity k_j .

We observe the case $R(z, w)$ has a maximally fixed algebraic function $a(z)$. First we suppose that $K_a(z)$ in (2.5) is a constant. Assume that $H_a(z_0, f(sz_0)) = 0$ for some z_0 . Then by (2.2) and (2.5), we have $H_a(z_0/s, f(z_0)) = 0$. Using (1.1) and (2.5), we get $H_a(z_0/s^2, f(z_0/s)) = 0$. Repeating this process, a meromorphic function $H_a(z, f(sz))$ has zeros $\{z_0/s^n\}$, $n = 1, 2, \dots$. This implies that $H_a(z, f(sz)) \equiv 0$, and hence $T(r, f) = O(\log r)$, a contradiction. Hence $a(z/s) \neq 0$ for any z , and hence $a(z/s)$ is a Picard function for $f(z)$.

Next we suppose that $K_a(z)$ in (2.5) is not a constant. By (2.2) and (2.5), if $K(z_0) = 0$, then $H_a(z_0, f(sz_0)) = 0$. Suppose that $K(z_0) = 0$ with multiplicity $\nu > 0$ and $K(s^k z_0) \neq 0$ for $k \geq 1$. Then by (2.2) and (2.5), we see that $s^n z_0$, $n = 1, 2, \dots$, are zeros of $H_a(z, f(sz))$ with multiplicity νd^n .

The observation to the case $R(z, w)$ has a maximally fixed pair of algebraic functions $c(z)$ and $d(z)$ is similarly discussed as above. By means of (2.2) and (2.5), (2.3), (2.4) and (1.1), $H_c(z, f(sz))$, $H_d(z, f(sz))$ can be 0 only for $s^n z_0$, $n \geq 0$, where z_0 is a zero of $K_c(z)$ or $K_d(z)$.

Let $a(z)$ be a maximally fixed algebraic function of $R(z, w)$ defined by $H(z, w) = 0$ with $\deg_w[H] = p$. Then obviously

$$\overline{N}(r, a; f) = \frac{1}{p} \overline{N}(r, 0; H(z, f(z))) = O((\log r)^2) = o(T(r, f))$$

though it may be $N(r, a; f) \neq o(T(r, f))$. For a maximally fixed pair of algebraic functions $c(z)$ and $d(z)$, similarly we get $\overline{N}(r, c; f) = o(T(r, f))$ and $\overline{N}(r, d; f) = o(T(r, f))$. Therefore, such functions $a(z)$, $c(z)$ and $d(z)$ are totally ramified Picard exceptional functions, including Picard functions.

We discuss numbers of maximally fixed algebraic functions and maximally fixed pairs of algebraic functions below.

LEMMA 3 *Let $a_j(z)$, $j = 1, 2, 3$ be meromorphic functions in a domain D . Let $R(z, w)$ be a rational function in z and w with $d = \deg_w[R(z, w)] \geq 2$. Suppose that there exist rational functions $K_j(z) \not\equiv 0$, $j = 1, 2, 3$ and meromorphic functions $b_j(z)$, $j = 1, 2, 3$ in D such that for arbitrary w and $z \in D$*

$$R(z, w) - a_j(z) = K_j(z) \frac{(w - b_j(z))^d}{Q_j(z, w)}, \quad j = 1, 2, 3, \quad (2.6)$$

where $Q_j(z, w)$, $j = 1, 2, 3$, are monic polynomials in w whose coefficients are rational functions. Then at least two functions $a_j(z)$ coincide.

Proof of Lemma 3 Assume that $a_j(z)$, $j = 1, 2, 3$ are distinct. It is easy to see that $Q_j(z, w)$ $j = 1, 2, 3$ coincide. By (2.6), we see that $b_j(z)$, $j = 1, 2, 3$ must be distinct.

Eliminating $Q_j(z, w)$, from (2.6), we get for arbitrary w and $z \in D$

$$F(z, w) = \frac{K_1(z)(w - a_1(z))^d - K_2(z)(w - a_2(z))^d}{b_1(z) - b_2(z)} - \frac{K_1(z)(w - a_1(z))^d - K_3(z)(w - a_3(z))^d}{b_1(z) - b_3(z)} = 0.$$

If $(\partial F / \partial w)(z, w) \not\equiv 0$, then w is written as a function of z , a contradiction. We arrange $F(z, w)$ as a polynomial in w with rational function coefficients. If $(\partial F / \partial w)(z, w) \equiv 0$, then all coefficients of $F(z, w)$ vanishes. Using the conditions from the coefficients of w^d , w^{d-1} and w^{d-2} , we have for $z \in D$

$$\frac{2d(a_1(z) - a_2(z))(a_1(z) - a_3(z))(b_2(z) - b_3(z))}{(b_1(z) - b_2(z))(b_1(z) - b_3(z))} = 0,$$

which yields a contradiction. Hence we have proved Lemma 3. \blacksquare

We consider the case $R(z, w)$ admits a maximally fixed algebraic function or a maximally fixed pair of algebraic functions. If they are rational functions, we can write $R(z, w)$ in the form (2.6). If they are multi-valued, we choose z_0 that is not a branch point, and choose a small domain D which contains z_0 . Thus, Lemma 3 implies that the number of maximally fixed algebraic function to $R(z, w)$ is at most two. If it is two, then both of them are rational functions. If it is multi-valued algebraic function, then it is two-valued. We also see that if there exists a maximally fixed pair of algebraic functions, then both of them are rational functions.

At the end of this section, we state the generalization of the second fundamental theorem of Nevanlinna to algebraic functions. Let $f(z)$ be a transcendental meromorphic function. As in [6, p. 136 and p. 138], we have the following inequality:

THEOREM A *Let $a^{[1]}(z), \dots, a^{[q]}(z)$ be algebraic functions defined by equations $H_j(z, w) = 0$, with $\deg_w[H_j] = p_j$, $1 \leq j \leq q$, of the form (1.3). For any $\epsilon > 0$, we have*

$$\begin{aligned} m(r, f) + \sum_{j=1}^q m(r, a^{[j]}; f) \\ = m(r, f) + \sum_{j=1}^q \frac{1}{p_j} m(r, 0; H_j(z, f(z))) \leq (2 + \epsilon)T(r, f), \end{aligned}$$

outside a set E_ϵ , depending on ϵ , of finite measure.

Recently, more precise result has been obtained by Yamanoi [8].

THEOREM B *Let $a^{[1]}(z), \dots, a^{[q]}(z)$ and $H_j(z, w)$, $1 \leq j \leq q$, be the same as above. For any $\epsilon > 0$, we have*

$$\begin{aligned} (q - 2 - \epsilon)T(r, f) &\leq \sum_{j=1}^q \frac{1}{p_j} \overline{N}(r, 0; H_j(z, f(z))) + o(T(r, f)) \\ &= \sum_{j=1}^q \overline{N}(r, a^{[j]}; f) + o(T(r, f)), \end{aligned}$$

outside a set E_ϵ , depending on ϵ , of finite measure.

Let $f(z)$ be a transcendental meromorphic solution of (1.1) and let $a(z) \in E_{\mathbb{P}}^*(f)$ be an algebraic function defined by $H(z, w) = 0$ with $\deg_w[H] = p$. In view of Theorem B, we see that p is at most two. Theorem B gives a short proof to the assertions on the numbers of maximally fixed algebraic functions and maximally fixed pairs of algebraic functions that we have shown above.

3. PROOF OF PROPOSITION 1

Proof of Proposition 1 Following [7, p. 158], consider the equation

$$f(sz) = (1 + z)f(z)^2, \quad s > 2,$$

and its solution

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{s^n}\right)^{2^{n-1}},$$

for which $n(r, 0; f) = 2^n - 1$ for $s^n \leq r < s^{n+1}$. Hence we have

$$N(r, 0; f) = (2^n - n - 1) \log s + (2^n - 1) \log(r/s^n), \quad s^n \leq r < s^{n+1}, \quad (3.1)$$

is sectionally linear in $\log r$, with $(dN/d \log r)_- = 2^{n-1} - 1 < (dN/d \log r)_+ = 2^n - 1$ at $r = s^n$.

First we show that $0 \in E_V$, namely $\Delta(f, 0) > 0$. To see this, we estimate $m(r_n^{[1]}, 0; f)$ where $r_n = s^n$.

$$\begin{aligned} m(r_n^{[1]}, 0; f) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta})|} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \prod_{k=1}^{n-1} \frac{1}{|1 + s^{n-k} e^{i\theta}|^{2^{k-1}}} \cdot \frac{1}{|1 + e^{i\theta}|^{2^{n-1}}} \cdot \prod_{\ell=n+1}^{\infty} \frac{1}{|1 + s^{-(\ell-n)} e^{i\theta}|^{2^{\ell-1}}} d\theta \\ &\geq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{\Pi_1 \cdot |1 + e^{i\theta}|^{2^{n-1}} \cdot \Pi_2} d\theta, \end{aligned} \quad (3.2)$$

where $\Pi_1 = \prod_{k=1}^{n-1} (1 + s^{n-k})^{2^{k-1}}$ and $\Pi_2 = \prod_{\ell=n+1}^{\infty} (1 + s^{-(\ell-n)})^{2^{\ell-1}}$. Obviously

$$\begin{aligned} \Pi_1 &\leq \prod_{k=1}^{n-1} ((1 + s)^{n-k})^{2^{k-1}} = \prod_{k=1}^{n-1} (1 + s)^{(n-k)2^{k-1}} \leq (1 + s)^{2^n - n - 1} \leq (1 + s)^{2^n}, \\ \log \Pi_2 &= \sum_{\ell=n+1}^{\infty} 2^{\ell-1} \log \left(1 + \frac{1}{s^{\ell-n}}\right) \leq (s^n/2) \sum_{\ell=n+1}^{\infty} (2/s)^{\ell} = \frac{2^n}{s-2}, \end{aligned}$$

where $M_1 > 1$ is a constant. Thus we get from (3.2), by $\log^+ a^n = n \log^+ a$,

$$m(r_n^{[1]}, 0; f) \geq 2^n \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{M|1 + e^{i\theta}|} d\theta, \quad r = s^n, \quad (3.3)$$

where $M = (1 + s)e^{1/(s-2)}$.

Therefore, by (3.1) and (3.3), we see that $m(r_n^{[1]}, 0; f) \neq o(T(r_n^{[1]}, 0; f))$, and hence 0 is a Valiron deficiency for $f(z)$.

Next we show that for sufficiently large s the Schröder function satisfies $0 \notin E_N(f)$, namely $\delta(f, 0) = 0$. We consider a sequence $r_n^{[2]} = s^{n+1/2}$. By (3.2)

$$m(r_n^{[2]}, 0; f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left(\prod_{k=1}^n A_k \cdot \prod_{k=2n+1}^{\infty} \left| 1 + \frac{s^{n+1/2}}{s^k} e^{i\theta} \right|^{-2^{k-1}} \right) d\theta, \quad (3.4)$$

where

$$A_k = \left| 1 + \frac{s^{n+1/2} e^{i\theta}}{s^k} \right|^{-2^{k-1}} \cdot \left| 1 + \frac{s^{n+1/2} e^{i\theta}}{s^{2n+1-k}} \right|^{-2^{2n-k}}.$$

We have $A_k \leq |s^{n+1/2-k} - 1|^{-2^{k-1}} \cdot |1 - s^{-(n+1/2-k)}|^{-2^{2n-k}}$, and for $1 \leq k \leq n$

$$\begin{aligned} -\log A_k &\geq 2^{k-1} \log(s^{n+1/2-k} - 1) + 2^{2n-k} \log(1 - s^{-(n+1/2-k)}) \\ &= 2^{n-1/2} (2^{-(n-k+1/2)} \log(s^{n-k+1/2} - 1) \\ &\quad + 2^{n-k+1/2} \log(1 - s^{-(n-k+1/2)})). \end{aligned}$$

For the sake of simplicity, we write $X = n - k + 1/2$. Then

$$\begin{aligned} -\log A_k &\geq 2^{n-1/2} (2^{-X} \log(s^X) + (2^{-X} + 2^X) \log(1 - s^{-X})) \\ &\geq 2^{n-1/2} (X 2^{-X} \log s - 2(2/s)^X - 2(2s)^{-X}) = 2^{n-1/2} G(X, s). \end{aligned} \quad (3.5)$$

We have that $G(X, s)$ is increasing in X and s . Hence we see that $G(X, s) > 0$, $X = 1/2, 3/2, \dots, (n-1)/2$ for sufficiently large s . It suffices to choose $s > 2^5$. Thus by (3.5) we get $A_k < 1$, $k = 1, 2, \dots, n$. Using the inequality $-\log(1-t) \leq 2t$, $0 < t < 1/2$, we get

$$\begin{aligned} m(r_n^{[2]}, 0; f) &\leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \prod_{k=2n+1}^{\infty} \left| 1 + \frac{s^{n+1/2}}{s^k} e^{i\theta} \right|^{-2^{k-1}} d\theta \\ &\leq \sum_{k=2n+1}^{\infty} s^{n+1/2} (2/s)^k = 2^n \left(\frac{2}{s} \right)^2 \cdot \frac{2s^{1/2}}{s-2} = o(2^n) = o(T(r, f)), \end{aligned}$$

as $n \rightarrow \infty$. Hence 0 is not a Nevanlinna deficiency, supposed that $s > 2^5$. ■

4. PROOF OF THEOREM 2

Proof of Theorem 2 We follow closely the method in [1]. Define

$$R^{*1}(z, w) = R(z, w), \quad \text{and} \quad R^{*n}(z, w) = R(z, R^{*(n-1)}(z/s, w)), \quad n \geq 2.$$

Let $a(z)$ be an algebraic function defined by $H(z, w) = 0$ and suppose that $a(z) \notin E_{\mathbb{P}}^*(f)$. We can write

$$H(z, R^{*3}(z, w)) = \frac{K_3(z)H_1(z/s, w)^{k_1} \cdots H_q(z/s, w)^{k_q}}{Q_3(z, w)}, \quad (4.1)$$

where $K_3(z) \in \mathbb{C}(z)$, $Q_3(z, w)$ is a polynomial in z and w with $\deg_w[Q_3] \leq d^3$ and $H_j(z/s, w)$, $j = 1, 2, \dots, q$, are relatively prime irreducible polynomials in w having rational function coefficients with $\deg_w[H_j] = p_j$. We assert that $q \geq 2$ in (4.1). If we assume the contrary, say assume that $q = 1$. Then by Lemma 3, and by the arguments in Section 2, we see that $a(z)$ is a maximally fixed algebraic function, or there exists a maximally fixed pair of rational functions in which one of them is $a(z)$. This implies that $a(z) \in E_{\mathbb{P}}^*(f)$, a contradiction. Then $p \leq p_j$ and $q \geq 2$. Hence $k_j \leq d^3 - 1$. That is, $c^{[j]}(z)$, defined by $H_j(z, w) = 0$ is of multiplicity at most $d^3 - 1$ in (4.1). Then obviously

$$p \cdot m(r, a; R^{*3}(z, f(z))) \leq \sum_{j=1}^q p_j \cdot k_j \cdot m(r, c^{[j]}; f(z)) + O(\log r). \quad (4.2)$$

Note that, in (4.2), $p_j k_j \leq p d^3$, hence $p_j \leq p d^3$. Let

$$H(z, R^{*3N}(z, w)) = \frac{K_{3N}(z) \cdot H_1^*(z/s, w)^{k_1^*} \cdots H_t^*(z/s, w)^{k_t^*}}{Q_{3N}(z, w)}, \quad (4.3)$$

and $\deg_w[H_j^*] = p_j^* \geq p$. Repeating this process, we have $k_j^* \leq (d^3 - 1)^N$, and

$$p \cdot m(r, a; f) \leq p \cdot d^3 \cdot (d^3 - 1)^N \sum_{j=1}^t m\left(\frac{r}{|s|^{3N}}, c^{*[j]}; f\right) + O(\log r), \quad (4.4)$$

where $c^{*[j]}(z)$ is defined by $H_j^*(z, w) = 0$ in (4.3). By means of Theorem A and (4.4) we obtain

$$p \cdot m(r, a; f) \leq p \cdot d^3 \cdot (2 + \epsilon)(d^3 - 1)^N T\left(\frac{r}{|s|^{3N}}, f\right) + O(\log r). \quad (4.5)$$

By the Valiron–Mohon'ko theorem and (1.1), we have

$$T\left(\frac{r}{|s|^{3N}}, f\right) = \frac{1}{d^{3N}} T(r, f) + O(\log r). \quad (4.6)$$

It follows from (4.5), (4.6) and (1.2) that

$$m(r, a; f) \leq d^3 \cdot (2 + \epsilon) \frac{K_2}{K_1} \left(\frac{d^3 - 1}{d^3} \right)^N T(r, f) + O(\log r).$$

Divide by $T(r, f)$ and let $r \rightarrow \infty$. Since N can be taken arbitrarily large, we have thus obtained (1.6) ■

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