RICCATI DIFFERENTIAL EQUATIONS WITH ELLIPTIC COEFFICIENTS II

Katsuya Ishizaki[1)], Ilpo Laine, Shun Shimomura[2)] and Kazuya Tohge[1)]

Abstract. We study a Riccati differential equation whose coefficient is expressible in terms of a special Weierstrass pe-function. We show that all the solutions are meromorphic, and examine the periodicity of them.

1. Introduction. In our preceding paper [4], we studied the Riccati differential equation

$$w' + w^2 + \frac{1}{4}(1 - m^2)\wp(0, g_3; z) = 0, \tag{1.1}$$

where

(1) m is a natural number such that $m \geq 2$, $m \notin 6\mathbb{N} = \{6n \mid n \in \mathbb{N}\}$;

(2) $\wp(0, g_3; z)$ is the Weierstrass $\wp$-function satisfying

$$(v')^2 = 4v^3 - g_3, \qquad g_3 \neq 0.$$

Let $\wp(z)$ be an arbitrary $\wp$-function satisfying $(v')^2 = 4v^3 - g_2 v - g_3$, $g_2^3 - 27g_3^2 \neq 0$. As was explained in [4, Section1], under a certain condition, if, for various values of a, an equation of the form $w' + w^2 + a\wp(z) = 0$ admits a plenty of meromorphic solutions, then it is either (1.1) or

$$w' + w^2 + \frac{1}{4}(1 - m^2)\wp_0(z) = 0, \tag{1.2}$$

where

(1) m is a natural number such that $m \geq 2$, $m \notin 4\mathbb{N} = \{4n \mid n \in \mathbb{N}\}$;

(2) $\wp_0(z) = \wp(g_2, 0; z)$ is the Weierstrass $\wp$-function satisfying

$$(v')^2 = 4v^3 - g_2 v, \qquad g_2 \neq 0. \tag{1.3}$$

Let ω_1^0, ω_2^0 be primitive periods of $\wp_0(z)$ satisfying $\mathrm{Im}(\omega_2^0/\omega_1^0) > 0$ (cf. (2.3)).

The main results of this paper are stated as follows.

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THEOREM 1.1. *All the solutions of* (1.2) *are meromorphic in the whole complex plane* $\mathbb{C}$.

THEOREM 1.2. *Suppose that* m *is even. Then,*

(i) *every solution of* (1.2) *is a doubly periodic function with periods* $(2\omega_1^0, 2\omega_2^0)$;

(ii) *there exist exactly two distinct solutions with periods* $(\omega_1^0, 2\omega_2^0)$ *(or with periods* $(2\omega_1^0, \omega_2^0)$*)*;

(iii) *there exists no solution with periods* (ω_1^0, ω_2^0).

THEOREM 1.3. *For every odd integer* m *satisfying* $m \geq 3$, *the equation* (1.2) *admits no periodic solution except a doubly periodic one, which is expressible in the form:*

$$\psi_m(z) = \frac{\wp_0'(z)}{2\wp_0(z)} + \sum_{h=1}^{k} \frac{2\wp_0(z)\wp_0'(z)}{\wp_0(z)^2 - \theta_{m,h}}, \qquad \text{if } m = 8k+3, \quad k = 0, 1, 2, ...,$$

$$\psi_m(z) = \frac{\wp_0''(z)}{\wp_0'(z)} - \frac{\wp_0'(z)}{2\wp_0(z)} + \sum_{h=1}^{k} \frac{2\wp_0(z)\wp_0'(z)}{\wp_0(z)^2 - \theta_{m,h}}, \qquad \text{if } m = 8k+5, \quad k = 0, 1, 2, ...,$$

$$\psi_m(z) = \frac{\wp_0''(z)}{\wp_0'(z)} + \sum_{h=1}^{k} \frac{2\wp_0(z)\wp_0'(z)}{\wp_0(z)^2 - \theta_{m,h}}, \qquad \text{if } m = 8k+7, \quad k = 0, 1, 2, ...,$$

$$\psi_m(z) = \sum_{h=1}^{k+1} \frac{2\wp_0(z)\wp_0'(z)}{\wp_0(z)^2 - \theta_{m,h}}, \qquad \text{if } m = 8k+9, \quad k = 0, 1, 2, ...,$$

where, for each (m, h), $\theta_{m,h}$ *is some complex constant.*

Using the properties of $\wp_0(z)$ explained in Section 2, we prove these results in Sections 3 and 4. For a related result concerning linear systems with doubly periodic coefficients, see [1].

2. Properties of the elliptic function $\wp_0(z)$. We review basic facts concerning elliptic functions (see [6], [7]). The elliptic function $\wp_0(z) = \wp(g_2, 0; z)$ satisfies (1.3), which is written in the form

$$(2.1) \qquad (v')^2 = 4v(v - e_1)(v - e_2),$$
$$e_1 = g_2^{1/2}/2, \quad e_2 = -g_2^{1/2}/2, \quad e_3 = 0, \quad g_2 \neq 0.$$

Consider the expression of $\wp_0(z)$:

$$(2.2) \qquad \wp_0(z) = \frac{1}{z^2} + \sum_{(p,q)\in\mathbb{Z}_*^2} \left(\frac{1}{(z - \Omega_{p,q})^2} - \frac{1}{\Omega_{p,q}^2} \right), \quad \mathbb{Z}_*^2 = \mathbb{Z}^2 - \{(0,0)\},$$

where $\Omega_{p,q} = p\omega_1^0 + q\omega_2^0$, $(p, q) \in \mathbb{Z}_*^2$ constitute the lattice of poles. By (2.1) the periods ω_1^0, ω_2^0 of $\wp_0(z)$ may be given by

$$(2.3) \qquad \omega_1^0 = \sqrt{2} g_2^{-1/4} \varepsilon_0, \quad \omega_2^0 = i\omega_1^0, \quad \varepsilon_0 = \int_{-1}^{0} \frac{dt}{\sqrt{t^3 - t}}.$$

Then we have

PROPOSITION 2.1. $\wp_0(\omega_j^0/2) = e_j$ $(j = 1, 2)$, $\wp_0(\omega_3^0/2) = 0$, *where* $\omega_3^0 = \omega_1^0 + \omega_2^0$.

Furthermore the Weierstrass ζ-function

$$\zeta_0(z) = \frac{1}{z} + \sum_{(p,q)\in\mathbb{Z}_*^2} \left(\frac{1}{z - \Omega_{p,q}} + \frac{1}{\Omega_{p,q}} + \frac{z}{\Omega_{p,q}^2} \right), \qquad \zeta_0'(z) = -\wp_0(z)$$

has the properties:

$$\zeta_0(z + \omega_j^0) = \zeta_0(z) + 2\eta_j^0, \quad j = 1, 2, 3, \tag{2.4}$$

$$\eta_j^0 = \zeta_0(\omega_j^0/2) = \frac{1}{\omega_j^0/2} + \sum_{(p,q)\in\mathbb{Z}_*^2} \left(\frac{1}{\omega_j^0/2 - \Omega_{p,q}} + \frac{1}{\Omega_{p,q}} + \frac{\omega_j^0/2}{\Omega_{p,q}^2} \right), \tag{2.5}$$

$$\eta_1^0\omega_2^0 - \eta_2^0\omega_1^0 = \pi i. \tag{2.6}$$

Relation (2.6) implies $(\eta_1^0, \eta_2^0) \neq (0, 0)$. Observing that $-i\Omega_{p,q} = \Omega_{q,-p}$, from (2.3) and (2.5), we obtain

PROPOSITION 2.2. $\eta_1^0/\eta_2^0 = \zeta_0(\omega_1^0/2)/\zeta_0(\omega_2^0/2) = i$.

Around each lattice pole $z = \sigma_L = \Omega_{p(L),q(L)}$, the Laurent series expansion of $\wp_0(z)$ is given by the following

PROPOSITION 2.3. *For an arbitrary pole* $z = \sigma_L$ *of* $\wp_0(z)$,

$$\wp_0(z) = \sum_{n=0}^{\infty} b_{4n}(z - \sigma_L)^{4n-2}, \quad b_0 = 1,$$

around $z = \sigma_L$.

PROOF. It sufficies to consider the case where $\sigma_L = 0$. We put $\wp_0(z) = \sum_{k=0}^{\infty} b_k z^{k-2}$, $b_0 = 1$, near $z = 0$. Then $-\wp_0(iz) = \sum_{k=0}^{\infty} i^k b_k z^{k-2}$. Since $-i\Omega_{p,q} = \Omega_{q,-p}$, we have $\wp_0(z) = -\wp_0(iz)$, which implies $b_k = 0$ for $k \notin 4\mathbb{N}$. □

Let $\varpi_0(z) = \wp_0(z)^{1/2}$ be a branch such that $\lim_{z\to 0} z\varpi_0(z) = 1$. Then $\varpi_0(z)$ is a doubly periodic function with the periods $(2\omega_1^0, \omega_3^0)$, which has two simple poles with residues 1 and -1 in its period parallelogram. A simple computation leads us to the following.

PROPOSITION 2.4. *The functions* $\varpi_0(z)$ *and* $W_0(z) = 2\varpi_0'(z) = \wp_0'(z)\wp_0(z)^{-1/2}$ *satisfy*

$$\varpi_0'(z)^2 = \varpi_0(z)^4 - g_2/4, \tag{2.7}$$

and

$$W_0''(z) = 6\wp_0(z)W_0(z), \tag{2.8}$$

respectively.

3. Proofs of Theorems 1.1 and 1.2. Consider the linear differential equation

$$u'' + \frac{1-m^2}{4}\wp_0(z)u = 0, \tag{3.1}$$

which is associated with (1.2).

LEMMA 3.1. *Let $z = \sigma_L$ be an arbitrary lattice pole of $\wp_0(z)$. Then (3.1) admits linearly independent solutions expressed in the form*

$$U_1(z) = (z-\sigma_L)^{(1-m)/2}\sum_{j=0}^{\infty}\beta_j^{(1)}(z-\sigma_L)^{4j}, \quad \beta_0^{(1)} = 1,$$
$$U_2(z) = (z-\sigma_L)^{(1+m)/2}\sum_{j=0}^{\infty}\beta_j^{(2)}(z-\sigma_L)^{4j}, \quad \beta_0^{(2)} = 1,$$

around $z = \sigma_L$.

PROOF. Around $z = \sigma_L$, we have

$$\wp_0(z) = (z-\sigma_L)^{-2}P_0((z-\sigma_L)^4)$$

with

$$P_0(t) = \sum_{n=0}^{\infty} b_{4n}t^n, \quad b_0 = 1$$

(cf. Proposition 2.3). Consider the equation

$$t^2\frac{d^2u}{dt^2} + \frac{3}{4}t\frac{du}{dt} + \frac{1-m^2}{64}P_0(t)u = 0 \tag{3.2}$$

around the regular singular point $t = 0$. The roots $\rho_1 = (1-m)/8$ and $\rho_2 = (1+m)/8$ of the indicial equation

$$\rho(\rho-1) + \frac{3}{4}\rho + \frac{1-m^2}{64} = 0$$

satisfy $\rho_2 - \rho_1 = m/4 \notin \mathbb{Z}$. Hence, (3.2) admits local solutions of the form

$$\varphi_1(t) = t^{(1-m)/8}\sum_{j=0}^{\infty}\beta_j^{(1)}t^j, \quad \varphi_2(t) = t^{(1+m)/8}\sum_{j=0}^{\infty}\beta_j^{(2)}t^j, \quad \beta_0^{(1)} = \beta_0^{(2)} = 1,$$

around $t = 0$ (see [2], [3]). By the transformation $t = (z-\sigma_L)^4$, (3.2) becomes (3.1) admitting the solutions $U_1(z) = \varphi_1((z-\sigma_L)^4)$, $U_2(z) = \varphi_2((z-\sigma_L)^4)$. This completes the proof. □

An arbitrary solution $w(z)$ of (1.2) is written in the form $w(z) = U'(z)/U(z)$, where $U(z)$ is a solution of (3.1). By Lemma 3.1, $w(z)$ is meromorphic in the whole complex plane $\mathbb{C}$, which completes the proof of Theorem 1.1.

Theorem 1.2 is proved by the same argument as that of the proof of [4, Theorem 3.1].

4. Proof of Theorem 1.3.

4.1. Case $m = 8k+3$. When $m = 8k+3$, $k = 0, 1, 2, \ldots$, we write (3.1) in the form

$$L^*(u) = 0, \quad L^* = (d/dz)^2 - (4k+1)(4k+2)\wp_0(z). \tag{4.1}$$

In what follows, $\wp_0(z)^{1/2}$ denotes the branch given in Section 2. Then we have

PROPOSITION 4.1. *For every $k \in \mathbb{N} \cup \{0\}$, (4.1) admits a doubly periodic solution of the form*

$$X_m(z) = \wp_0(z)^{1/2} \prod_{h=1}^{k} (\wp_0(z)^2 - \theta_{m,h})$$

($m = 8k+3$) with periods $(2\omega_1^0, \omega_3^0)$.

PROOF. Let $\Delta_0^{1/2}$ be the period parallelogram of $\wp_0(z)^{1/2}$ with vertices $(-\omega_1^0 - \omega_3^0)/2$, $(3\omega_1^0 - \omega_3^0)/2$, $(-\omega_1^0 + \omega_3^0)/2$ and $(3\omega_1^0 + \omega_3^0)/2$. The poles of $\wp_0(z)^{1/2}$ in $\Delta_0^{1/2}$ are $z = 0$ and $z = \omega_1^0$, whose residues are 1 and -1, respectively. By Proposition 2.3, for every $q \in \mathbb{N} \cup \{0\}$, we have

$$\wp_0(z)^{1/2}\wp_0(z)^q = z^{-2q-1} \sum_{n=0}^{\infty} b_{4n}^{(q)} z^{4n}, \qquad b_0^{(q)} = 1 \tag{4.2.1}$$

around $z = 0$, and

$$\wp_0(z)^{1/2}\wp_0(z)^q = -(z-\omega_1^0)^{-2q-1} \sum_{n=0}^{\infty} b_{4n}^{(q)} (z-\omega_1^0)^{4n} \tag{4.2.2}$$

around $z = \omega_1^0$. Then, for $\nu = 0, 1, \ldots, k-1, k$,

$$\begin{aligned} L^*(\wp_0(z)^{1/2}\wp_0(z)^{2\nu}) &= z^{-4\nu-3} \sum_{n=0}^{\infty} B_{4n}^{\nu,k} z^{4n} \\ &= B_0^{\nu,k} z^{-4\nu-3} + B_4^{\nu,k} z^{-4\nu+1} + \cdots + B_{4\nu}^{\nu,k} z^{-3} + O(z), \end{aligned}$$

where $B_0^{\nu,k} = (4\nu+1)(4\nu+2) - (4k+1)(4k+2)$. Observing that $B_0^{k,k} = 0$, $B_0^{\nu,k} \neq 0$ $(\nu \neq k)$, we can choose $C_{k,n} \in \mathbb{C}$, $n = 0, 1, \ldots, k$, satisfying $C_{k,k} = 1$ in such a way that

$$L^*\left(\wp_0(z)^{1/2} \sum_{n=0}^{k} C_{k,n}\wp_0(z)^{2n}\right) = O(z)$$

near $z = 0$. Then, by (4.2.1) and (4.2.2),

$$L^*\left(\wp_0(z)^{1/2} \sum_{n=0}^{k} C_{k,n}\wp_0(z)^{2n}\right) = O(z - \omega_1^0)$$

also holds near $z=\omega_1^0$. By this fact and the Liouville theorem, we conclude that

$$X_m(z)=\wp_0(z)^{1/2}\sum_{n=0}^{k}C_{k,n}\wp_0(z)^{2n}=\wp_0(z)^{1/2}\prod_{h=1}^{k}(\wp_0(z)^2-\theta_{m,h})$$

satisfies

$$L^*(X_m(z))\equiv 0,$$

which implies the proposition. □

It is easy to see that

$$\psi_m(z)=\frac{X'_m(z)}{X_m(z)}=\frac{\wp'_0(z)}{2\wp_0(z)}+\sum_{h=1}^{k}\frac{2\wp_0(z)\wp'_0(z)}{\wp_0(z)^2-\theta_{m,h}} \tag{4.3}$$

is a solution of (1.2).

In order to verify that there exists no periodic solution of (1.2) other than $\psi_m(z)$, we examine another solution of (4.1). By the uniqueness of the solution of an initial value problem associated with (4.1), every zero of $X_m(z)$ is simple. Hence each constant $\theta_{m,h}$ satisfies $\theta_{m,h}\neq 0$, $\theta_{m,h}\neq\theta_{m,i}$ for $i\neq h$. It is easy to see that all zeros are located symmetrically with respect to $z=0$. Furthermore, $X_m(z)^2$ is a doubly periodic function with periods (ω_1^0,ω_2^0). It follows from these facts and the Liouville theorem, that

$$\frac{1}{X_m(z)^2}=\frac{1}{2}\sum_{\tau\in Z}\frac{1}{X'_m(\tau)^2}\Big(\wp_0(z-\tau)+\wp_0(z+\tau)-2\wp_0(\tau)\Big),$$

where Z denotes the set of all zeros of $X_m(z)$ in

$$\Delta_0=\big\{s_1\omega_1^0+s_2\omega_2^0\;\big|\;-1/2<s_1\le 1/2,\;-1/2<s_2\le 1/2\big\}. \tag{4.4}$$

Then we have another solution of (4.1) written in the form

$$Y_m(z)=X_m(z)\int_{z_0}^{z}\frac{dt}{X_m(t)^2}=-\frac{X_m(z)}{2}\sum_{\tau\in Z}\frac{1}{X'_m(\tau)^2}\Big(\zeta_0(z-\tau)+\zeta_0(z+\tau)+2\wp_0(\tau)z\Big)$$

(see Section 2). For the linearly independent solutions $X_m(z)$ and $Y_m(z)$, we have the Floquet matrices

$$M_j=\begin{pmatrix}1&\delta_j\\0&1\end{pmatrix},\quad \delta_j=-\omega_j^0\sum_{\tau\in Z}\frac{\wp_0(\tau)}{X'_m(\tau)^2}-2\eta_j^0\sum_{\tau\in Z}\frac{1}{X'_m(\tau)^2}\quad (j=1,2),$$

satisfying $[\omega_j^0](X_m(z),Y_m(z))=(X_m(z),Y_m(z))M_j$, where $[\omega_j^0]$ denotes the analytic continuation along the segment $[z,z+\omega_j^0]$ (cf. Sections 2 and [4, Section3]). Note that Z is written in the form

$$Z=Z_0\cup\Big(\bigcup_{h=1}^{k}Z_h\Big)$$

with

$$Z_0=\big\{\tau\in Z\;\big|\;\wp_0(\tau)=0\big\},$$
$$Z_h=\big\{\pm\tau_{h,-},\pm\tau_{h,+}\in Z\;\big|\;\wp_0(\pm\tau_{h,-})=-\theta_{m,h}^{1/2},\;\wp_0(\pm\tau_{h,+})=\theta_{m,h}^{1/2}\big\}.$$

LEMMA 4.2. *We have*

$$\sum_{\tau \in Z} \frac{\wp_0(\tau)}{X_m'(\tau)^2} = 0.$$

PROOF. Since every zero of $X_m(z)$ is simple,

$$\sum_{\tau \in Z_0} \frac{\wp_0(\tau)}{X_m'(\tau)^2} = 0. \tag{4.5}$$

Observe that

$$\begin{aligned} X_m'(\tau_{h,\pm})^2 &= \wp_0(\tau_{h,\pm}) \cdot 4\wp_0'(\tau_{h,\pm})^2 \wp_0(\tau_{h,\pm})^2 \prod_{q \neq h} \big(\wp_0(\tau_{h,\pm})^2 - \theta_{m,q}\big)^2 \\ &= 4\wp_0(\tau_{h,\pm})^4 \big(4\wp_0(\tau_{h,\pm})^2 - g_2\big) \prod_{q \neq h} \big(\wp_0(\tau_{h,\pm})^2 - \theta_{m,q}\big)^2 \\ &= 4\theta_{m,h}^2 (4\theta_{m,h} - g_2) \prod_{q \neq h} \big(\theta_{m,h} - \theta_{m,q}\big)^2 = \Gamma_h \neq 0, \end{aligned}$$

and that

$$X_m'(-\tau_{h,\pm})^2 = \Gamma_h \neq 0.$$

Hence we have

$$\begin{aligned} &\sum_{\tau \in Z_h} \frac{\wp_0(\tau)}{X_m'(\tau)^2} \\ &\qquad = \Gamma_h^{-1} \Big(\big(\wp_0(\tau_{h,-}) + \wp_0(\tau_{h,+})\big) + \big(\wp_0(-\tau_{h,-}) + \wp_0(-\tau_{h,+})\big) \Big) = 0. \end{aligned} \tag{4.6}$$

From (4.5) and (4.6), the lemma immediately follows. □

By Lemma 4.2, we have $\delta_j = -2\eta_j^0 \sum_{\tau \in Z} X_m'(\tau)^{-2}$ ($j = 1, 2$), which satisfy $(\delta_1, \delta_2) \neq (0, 0)$. Indeed, if $\delta_1 = \delta_2 = 0$, then (4.1) admits a nontrivial doubly periodic solution which vanishes at each pole of $\wp_0(z)$, contradicting the Liouville theorem. Let δ be the ratio

$$\delta = \begin{cases} \delta_1/\delta_2, & \text{if } \delta_2 \neq 0, \\ 0, & \text{if } \delta_2 = 0. \end{cases}$$

Now, we note the following criteria, which is proved by the same way as in the proof of [4, Proposition 4.5].

LEMMA 4.3. *If* $\delta \notin \mathbb{Q}$, *then there exists no periodic solution of* (1.2) *other than* (4.3). *If* $\delta \in \mathbb{Q}$, *then every solution of* (1.2) *is purely simply periodic.*

Since $\sum_{\tau \in Z} X_m'(\tau)^{-2} \neq 0$, using Proposition 2.2, we have

$$\delta = \eta_1^0 / \eta_2^0 = i. \tag{4.7}$$

Hence, by Lemma 4.3, there exists no periodic solution other than (4.3).

4.2. Case $m = 8k+7$. When $m = 8k+7$, $k = 0, 1, 2, \ldots$, we can construct a solution of (3.1) expressible in the form

$$\tilde{X}_m(z) = \wp_0'(z)\Big(\wp_0(z)^{2k} + \sum_{n=0}^{k-1} \tilde{C}_n \wp_0(z)^{2n}\Big) = \wp_0'(z) \prod_{h=1}^{k} \big(\wp_0(z)^2 - \theta_{m,h}\big),$$

by an argument analogous to that for the case $m = 8k+3$ (see also [4, Section 4]). Then

$$\psi_m(z) = \frac{\tilde{X}_m'(z)}{\tilde{X}_m(z)} = \frac{\wp_0''(z)}{\wp_0'(z)} + \sum_{h=1}^{k} \frac{2\wp_0(z)\wp_0'(z)}{\wp_0(z)^2 - \theta_{m,h}} \tag{4.8}$$

is a periodic solution of (1.2). By the same argument as in Section 4.1, we obtain the Floquet matrices

$$\tilde{M}_j = \begin{pmatrix} 1 & \tilde{\delta}_j \\ 0 & 1 \end{pmatrix}, \quad \tilde{\delta}_j = -\omega_j^0 \sum_{\tau \in \tilde{Z}} \frac{\wp_0(\tau)}{\tilde{X}_m'(\tau)^2} - 2\eta_j^0 \sum_{\tau \in \tilde{Z}} \frac{1}{\tilde{X}_m'(\tau)^2}, \quad (j = 1, 2),$$

where $\tilde{Z}$ denotes the set of all zeros of $\tilde{X}_m(z)$ in Δ_0 (cf. (4.4)). Decompose the set $\tilde{Z}$ into

$$\begin{aligned} \tilde{Z} &= \tilde{Z}' \cup \Big(\bigcup_{h=1}^{k} \tilde{Z}_h\Big), \\ \tilde{Z}' &= \big\{\tau \mid \wp_0'(\tau) = 0\big\} = \{\omega_1^0/2, \omega_2^0/2, \omega_3^0/2\}, \\ \tilde{Z}_h &= \big\{\tau \mid \wp_0(\tau)^2 = \theta_{m,h}\big\}. \end{aligned} \tag{4.9}$$

Using the formulas

$$\begin{aligned} \wp_0(\omega_1^0/2) = {} & g_2^{1/2}/2, \quad \wp_0(\omega_2^0/2) = -g_2^{1/2}/2, \quad \wp_0(\omega_3^0/2) = 0, \\ \tilde{X}_m'(\omega_j^0/2)^2 &= \wp_0''(\omega_j^0/2)^2 \prod_{h=1}^{k} \Big(\wp_0(\omega_j^0/2)^2 - \theta_{m,h}\Big)^2 \\ &= g_2^2 \prod_{h=1}^{k} (g_2/4 - \theta_{m,h})^2 \qquad (j = 1, 2), \end{aligned}$$

we have

$$\sum_{\tau \in \tilde{Z}'} \frac{\wp_0(\tau)}{\tilde{X}_m'(\tau)^2} = 0. \tag{4.10}$$

Furthermore, by the same argument as in the proof of Lemma 4.2, we have

$$\sum_{\tau \in \tilde{Z}_h} \frac{\wp_0(\tau)}{\tilde{X}_m'(\tau)^2} = 0 \qquad (h = 1, \ldots, k). \tag{4.11}$$

From (4.9), (4.10), (4.11) and Proposition 2.2, it follows that $\tilde{\delta} = \tilde{\delta}_1/\tilde{\delta}_2 = i$. Hence, applying Lemma 4.3, we conclude that there exists no periodic solution of (1.2) other than (4.8).

4.3. Cases $m = 8k+5$ and $m = 8k+9$. When $m = 8k+5$, (3.1) is written in the form

$$L_k(u) = 0, \quad L_k = (d/dz)^2 - (4k+2)(4k+3)\wp_0(z). \tag{4.12}$$

Then we have

PROPOSITION 4.4. *For every $k = 0, 1, 2, \ldots$, (4.12) admits a solution expressed as*

$$W_k(z) = \wp_0'(z)\wp_0(z)^{-1/2}\sum_{n=0}^{k}\tilde{C}_{k,n}\wp_0(z)^{2n} \tag{4.13}$$

with $\tilde{C}_{k,k} = 1$.

Proof. We show the conclusion by induction on k. By (2.8) the function $W_0(z) = \wp_0'(z)\wp_0(z)^{-1/2}$ satisfies (4.12) with $k = 0$. Suppose that, for $k = 0, 1, \ldots, \kappa - 1$, (4.12) admits a solution expressed as (4.13). By Proposition 4.1, for suitably chosen constants C_n, $n = 0, 1, \ldots, \kappa$, the function

$$X(z) = \wp_0(z)^{1/2}\sum_{n=0}^{\kappa}C_n\wp_0(z)^{2n}, \quad C_\kappa = 1$$

satisfies

$$X''(z) = (4\kappa+1)(4\kappa+2)\wp_0(z)X(z). \tag{4.14}$$

Differentiate (4.14) and put $w_\kappa(z) = X'(z)$. Observing that

$$\frac{\wp_0'(z)}{\wp_0(z)}X(z) - \frac{w_\kappa(z)}{2\kappa+1/2} = \wp_0'(z)\wp_0(z)^{-1/2}\sum_{n=0}^{\kappa-1}C_n'\wp_0(z)^{2n}, \quad C_n' \in \mathbb{C},$$

we have

$$L_\kappa(w_\kappa(z)) = (4\kappa+2)(4\kappa+3)\wp_0(z)\left(\wp_0'(z)\wp_0(z)^{-1/2}\sum_{n=0}^{\kappa-1}C_n''\wp_0(z)^{2n}\right), \quad C_n'' \in \mathbb{C}.$$

By supposition,

$$L_\kappa\big(w_\kappa(z) + \gamma_{\kappa-1}W_{\kappa-1}(z)\big) = L_\kappa(w_\kappa(z)) + \gamma_{\kappa-1}\rho_{\kappa,\kappa-1}\wp_0(z)W_{\kappa-1}(z),$$
$$\rho_{\kappa,\kappa-1} = (4\kappa+2)(4\kappa+3) - (4\kappa-2)(4\kappa-1) \neq 0.$$

Hence, if $\gamma_{\kappa-1} = -(4\kappa+2)(4\kappa+3)C''_{\kappa-1}/\rho_{\kappa,\kappa-1}$, then

$$L_\kappa\big(w_\kappa(z)+\gamma_{\kappa-1}W_{\kappa-1}(z)\big) = (4\kappa+2)(4\kappa+3)\wp_0(z)\left(\wp_0'(z)\wp_0(z)^{-1/2}\sum_{n=0}^{\kappa-2}C_n^{(3)}\wp_0(z)^{2n}\right).$$

Repeating this procedure, we may choose γ_j $(j = 0, ..., \kappa - 1)$ in such a way that $W_\kappa(z) = w_\kappa(z) + \sum_{j=0}^{\kappa-1} \gamma_j W_j(z)$ satisfies (4.12) with $k = \kappa$. Thus the proposition is verified. □

We can write (4.13) in the form

$$W_k(z) = \wp_0'(z)\wp_0(z)^{-1/2} \prod_{h=1}^{k} (\wp_0(z)^2 - \theta_{m,h}),$$

which yields the solution $\psi_m(z) = W_k'(z)/W_k(z)$ of (1.2) with $m = 8k + 5$.

Next consider the case where $m = 8k + 9$, $k = 0, 1, 2, ...$ It is easy to see that $V_0(z) = 6\wp_0(z)^2 - 9g_2/10$ satisfies

$$V_0''(z) = 20\wp_0(z)V_0(z),$$

which means that $V_0(z)$ is a solution of (3.1) with $m = 9$. Using this fact, from the solution of (3.1) with $m = 8k + 7$ given in Section 4.2, we can derive a solution of (3.1) with $m = 8k + 9$ written in the form

$$V_k(z) = \prod_{h=1}^{k+1} (\wp_0(z)^2 - \theta_{m,h}),$$

by the same argument as in the proof of Proposition 4.4. Then, $\psi_m(z) = V_k'(z)/V_k(z)$ is a doubly periodic solution of (1.2) with $m = 8k + 9$. Furthermore, in both cases $m = 8k + 5$ and $m = 8k + 9$, we can also verify the non-existence of periodic solutions of (1.2) other than $\psi_m(z)$ by the same way as in Section 4.1. This completes the proof.

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Department of Mathematics
Nippon Institute of Technology
4-1 Gakuendai Miyashiro
Minamisaitama Saitama 345-8501
Japan
E-mail address: ishi@nit.ac.jp

Department of Mathematics
Joensuu University
P.O.Box 111
FIN–80101 Joensuu
Finland
E-mail address: laine@ns.joensuu.fi

Department of Mathematics
Keio University
3-14-1 Hiyoshi Kohoku-ku
Yokohama 223-8522
JAPAN
E-mail address: shimomur@math.keio.ac.jp

Faculty of Technology
Kanazawa University
2-40-20 Kodatsuno
Kanazawa 920-8667
JAPAN
E-mail address: tohge@t.kanazawa-u.ac.jp