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Interrelations between difference equations and differential equations in complex domains

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複素領域における差分方程式と微分方程式の相互関係ついて

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ABSTRACT

We are concerned with differential equations and discrete functional equations in complex domains. Considering the existence of transcendental meromorphic solutions, we discuss interrelations between difference equations and differential equations mainly in the whole complex plane. We also treat linear difference equations of second order in connection with difference Riccati equations. Some examples are given.

要　旨

複素領域において，微分方程式と離散関数方程式を取り扱う。複素平面全体における超越的有理形関数解の存在・非存在を考慮して，差分方程式と微分方程式の相互関係ついて考察する。また，線形２階差分方程式と差分リッカチ方程式を結びつける方法を紹介する。本論文で紹介した議論に即した非自明な例を構成する。

1　Introduction

The theory of complex differential equations and the theory of complex discrete functional equations have been developed by giving impacts and influences each other with the remarkable developments of complex analysis. In fact, researches of algebraic differential equations and complex oscillation theory have been evolved by virtues of the Nevanlinna theory and the Wiman-Valiron theory, see e.g., [16], [19]. The considerations of the counterparts of these researches have required the constructions of discrete version of the value distribution theory of meromorphic functions. Here 'meromorphic' means that 'meromorphic whole complex plane'. On the other hand, the results of the discrete version of value distribution theory have been supported and corroborated by the discrete functional equations, for examples, difference equations. The properties of some complex analysis are indicated by the specific functions produced from the functional equations. During in the last decay, the progress of difference analogues of the Nevanlinna theory have advanced, e.g., [4], [8], [9], and the Wiman-Valiron theory has been generalized for hyperbolic domains, e.g., [2]. The difference analogues of the Wiman-Valiron theory were constructed and have been applied to built the counterparts of the theory of linear differential equations in the complex plane, e.g., [5], [15].

The Malmquist-Yosida theorem in the theory of complex differential equation states that

$$w' = P(z, w), \tag{1}$$

where $P(z, w)$ is a polynomial in w with rational coefficients, has no transcendental meromorphic solution when $\deg_w P(z, w) \geq 3$, [20], [27]. The corresponding difference equation to (1) seems to be

$$w(z+1) = P(z, w), \tag{2}$$

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where $P(z, w)$ is a polynomial in w with rational coefficients similar to (1). The counterpart of the Malmquist–Yosida theorem was proved by Yanagihara [26]. The difference equation (2) has no transcendental meromorphic solution of finite order when $\deg_w P(z, w) \geq 2$. The differential equation (1) of degree $\deg_w P(z, w) = 2$ is called Riccati equation, which has a transcendental meromorphic solution under some conditions. Riccati equation has been investigated in the complex plane from many aspects, e.g., [1], [11], [19], [24]. By virtue of the Yanagihara theorem, a relating difference equation to Riccati equation may be the difference equation (2) of degree $\deg_w P(z, w) = 1$. Considering the analytic properties of meromorphic solutions, the polynomial in (2) could be generalized to a rational function in w of degree 1 with meromorphic coefficients, namely

$$(3) \quad w(z+1) = \frac{a(z) + b(z)w}{c(z) + d(z)w}, \quad a(z)d(z) - b(z)c(z) \not\equiv 0,$$

where $a(z)$, $b(z)$, $c(z)$ and $d(z)$ are meromorphic functions. By suitable Möbius transformation $f(z) = M(z, w(z))$ with meromorphic coeffcients, (3) is reduced to a linear difference equation of first order, a difference equation $f(z+1)f(z) = \alpha(z)$, or

$$(4) \qquad f(z+1) = \frac{A(z) + f(z)}{1 - f(z)},$$

where $\alpha(z) \not\equiv 0$ and $A(z) \not\equiv -1$ are meromorphic functions concretely represented by $a(z)$, $b(z)$, $c(z)$ and $d(z)$. We call the difference equation (4) the difference Riccati equation in this paper. Recent results on (4) are found in, e.g., [3], [12], [13].

2 Continuous limit and gauge transformation

Concerning the interrelations between solutions of difference equations and solutions of differential equation, we first discuss the bilinear method to derive a difference equation $\omega_0 = \omega_0(z, f(z), f(z+1), \dots, f(z+k)) = 0$ from an algebraic differential equation $\omega_1 = \omega_1(z, f(z), f'(z), \dots, f^{(k)}(z)) = 0$, where k is a positive integer, see e.g., [6]. Set $f(z) = u(z)/v(z)$ in $\omega_1 = 0$. It is known that any algebraic differential equation is gauge invariant. In other words, for any $h(z)$, $\tilde{u}(z) = u(z)h(z)$ and $\tilde{v}(z) = v(z)h(z)$ also satisfy the same differential equation in place of $u(z)$ and $v(z)$ respectively. We note that in order to propose $\omega_0 = 0$ if we simply change $f(z+j)$, $j = 1, 2, \dots, k$ in place of $f^{(j)}$, $j = 1, 2, \dots, k$ in $\omega_1 = 0$, it does not always work well. To to this, we may choose a difference equation having the property of gauge invariant.

On the other hand, we have a method "continuous limit" to derive a differential equation from a difference equation, which has been contributed to Painlevé analysis, e.g., [7, §50], [21], [22], [23]. A rough sketch of this idea is the following. Let k be a positive integer, and ε be a complex number. We set a pair of relations $\mu(z, t, \varepsilon) = 0$ and $\nu(f(z), w(t, \varepsilon), \varepsilon) = 0$. According to these relations, we transform a difference equation $\Omega_0(z, f(z), f(z+1), \dots, f(z+k)) = 0$ to a certain difference equation $\Omega_1(t, w(t, \varepsilon), w(t+\varepsilon, \varepsilon), \dots, w(t+k\varepsilon, \varepsilon)) = 0$. Letting $\varepsilon \to 0$, with some conditions on coefficients of Ω_1, we derive a differential equation $\Omega_2(t, w(t, 0), w'(t, 0), w''(t, 0), \dots, w^{(k)}(t, 0)) = 0$.

Example 2.1 We consider an algebraic differential equation

$$(5) \qquad (w')^2 = A(z)(w^2 - 1),$$

where $A(z)$ is a meromorphic function. The author treated (5) paying attention to two distinct transcendental meromorphic solutions $w_1(z)$ and $w_2(z)$ when $A(z)$ is a rational function in [14]. It was shown that $w_1(z)$ and $w_2(z)$ satisfy an algebraic relation

$$w_1^2 + 2cw_1w_2 + w_2^2 = 1 - c^2,$$

where c is a constant. It is a curious problem whether the difference analogue of this property holds or not. Before we consider this problem, we should obtain the corresponding difference equation to (5). Here we choose a difference equation

$$(6) \qquad (\Delta f(z))^2 = A(z)(f(z)f(z+1) - 1),$$

where $\Delta f(z) = f(z+1) - f(z)$, and show that (6) is gauge invariant below. Moreover, we confirm that (6) reduces to (5) by continuous limit.

Set $f(z) = u(z)/v(z)$ in (6). Then we have

$$(7) \qquad \Big(u(z+1)v(z) - u(z)v(z+1)\Big)^2 = A(z)\Big(u(z)u(z+1)v(z)v(z+1) - v(z)^2 v(z+1)^2\Big).$$

Let $h(z) \not\equiv 0$ be an arbitrary function. Further we set $\tilde{u}(z) = u(z)h(z)$ and $\tilde{v}(z) = v(z)h(z)$ in (7). Multiplying $h(z)^4$ both side, we see that $\tilde{u}(z)$ and $\tilde{v}(z)$ satisfy (7), which implies that (6) is gauge invariant.

Setting $t = \varepsilon z$ and $f(z) = w(t, \varepsilon)$ in (6) and $\varepsilon^2 \tilde{A}(t, \varepsilon)$ in place of $A(z)$, we show that (6) reduces to (5).

Since $f(z+1) = w(\varepsilon(z+1), \varepsilon) = w(\varepsilon z + \varepsilon, \varepsilon) = w(t + \varepsilon, \varepsilon)$, we have

$$(8) \quad (w(t+\varepsilon, \varepsilon) - w(t, \varepsilon))^2 = \varepsilon^2 \tilde{A}(t, \varepsilon)(w(t, \varepsilon)w(t+\varepsilon, \varepsilon) - 1).$$

Assume that $\lim_{\varepsilon\to 0}\tilde{A}(t,\varepsilon)=\tilde{A}(t,0)$ exists. Letting $\varepsilon\to 0$ in (8), we see that $w(t,0)=\lim_{\varepsilon\to 0} w(t,\varepsilon)$, if exists, satisfies the differential equation

$$(9)\qquad w'(t)^2=\tilde{A}(t)(w(t)^2-1),$$

with $\tilde{A}(t)=\tilde{A}(t,0)$, which is of the form (5). The problem whether distinct meromorphic solutions $f_1(z)$ and $f_2(z)$ to (6) have some algebraic relation is most generally open.

3 Relations between linear difference equations and difference Riccati equations

Let $n\geq 2$ be an integer. We denote by $\mathfrak{C}(f_1, f_2, ..., f_n)(z)$ the Casoratian of functions $f_1(z), f_2(z), ..., f_n(z)$.

$$\mathfrak{C}(f_1,f_2,\dots,f_n)(z)$$
$$=\begin{vmatrix} f_1(z) & f_2(z) & \cdots & f_n(z)\\ \Delta f_1(z) & \Delta f_2(z) & \cdots & \Delta f_n(z)\\ \Delta^2 f_1(z) & \Delta^2 f_2(z) & \cdots & \Delta^2 f_n(z)\\ \vdots & \vdots & \ddots & \vdots\\ \Delta^{(n-1)} f_1(z) & \Delta^{(n-1)} f_2(z) & \cdots & \Delta^{(n-1)} f_n(z)\end{vmatrix}$$
$$=\begin{vmatrix} f_1(z) & f_2(z) & \cdots & f_n(z)\\ f_1(z+1) & f_2(z+1) & \cdots & f_n(z+1)\\ f_1(z+2) & f_2(z+2) & \cdots & f_n(z+2)\\ \vdots & \vdots & \ddots & \vdots\\ f_1(z+n-1) & f_2(z+n-1) & \cdots & f_n(z+n-1)\end{vmatrix}.$$

We consider a linear difference equation of second order $\mathfrak{C}(u, u_1, u_2)(z)=0$, i.e.,

$$(10)\qquad a_2(z)u(z+2)+a_1(z)u(z+1)+a_0(z)u(z)=0$$

with

$$(11)\qquad a_2(z)=\mathfrak{C}(u_1,u_2)(z),$$

$$(12)\qquad a_1(z)=-\begin{vmatrix} u_1(z) & u_2(z)\\ u_1(z+2) & u_2(z+2)\end{vmatrix},$$

$$(13)\qquad a_0(z)=\mathfrak{C}(\Delta u_1,\Delta u_2)(z).$$

Clearly, (10) possesses solutions $u_1(z)$ and $u_2(z)$. Assume $a_2(z)\not\equiv 0$ and set $u(z)=b(z)y(z)$ in (10). Using $\Delta^2 y(z)=y(z+2)-2y(z+1)+y(z)$, we obtain a linear difference equation

$$(14)\qquad \Delta^2 y(z)+\left(\frac{a_0(z)}{a_2(z)}\frac{b(z)}{b(z+2)}-1\right)y(z)=0$$

if $b(z)$ satisfies a difference equation

$$(15)\qquad b(z+1)=-\frac{1}{2}\frac{a_1(z-1)}{a_2(z-1)}b(z).$$

If $b(z)\not\equiv 0$ and $\mathfrak{C}(y_1, y_2)(z)\not\equiv 0$, where $y_j(z)=u_j(z)/b(z)$, $j=1, 2$, then any solution $y(z)$ of (14) can be represented

$$(16)\qquad y(z)=Q_1(z)y_1(z)+Q_2(z)y_2(z),$$

where $Q_j(z)$, $j=1, 2$ are periodic function of period 1. It is known that $f(z)=-\Delta y(z)/y(z)$ solves a difference Riccati equation (4) with

$$(17)\qquad A(z)=\frac{a_0(z)}{a_2(z)}\frac{b(z)}{b(z+2)}-1.$$

We note that by (15), $A(z)$ in (10) can be written as

$$(18)\qquad A(z)=4\frac{a_0(z)a_2(z-1)}{a_1(z)a_1(z-1)}-1.$$

In fact, by (15),

$$(19)\qquad \begin{aligned} f(z)&=-\frac{\Delta y(z)}{y(z)}=-\frac{\Delta(u(z)/b(z))}{u(z)/b(z)}\\ &=1-\frac{b(z)}{b(z+1)}\frac{u(z+1)}{u(z)}\\ &=1+2\frac{a_2(z-1)}{a_1(z-1)}\frac{u(z+1)}{u(z)}.\end{aligned}$$

Remark 3.1 It is known that if $a_1(z-1)/a_2(z-1)$ is a meromorphic function of finite order ρ then there exists a meromorphic solution to (15) of order at most $\rho+1$, see [25, Page 30, Theorem 5]. In case $a_1(z-1)/a_2(z-1)$ is a rational function, we obtain a meromorphic solution of (15) concretely by a formula, e.g., [17, Page 48], [18, Pages 115–116].

Example 3.1 We consider the Euler Γ-function $\Gamma(z)$. Set $\gamma(z)=1/\Gamma(z)$. It is known that $\Gamma(z)$ and $\gamma(z)$ satisfy difference equations of first order $\Gamma(z+1)=z\Gamma(z)$ and $\gamma(z+1)=\gamma(z)/z$, respectively. We set $u_1(z)=\Gamma(z)$ and $u_2(z)=\gamma(z)$ in (11), (12), and (13). Then

$$\begin{aligned} a_0(z)&=\left(\frac{1}{z+1}-(z+1)\right)\Gamma(z)\gamma(z),\\ a_1(z)&=-\left(\frac{1}{z(z+1)}-z(z+1)\right)\Gamma(z)\gamma(z),\\ a_2(z)&=\left(\frac{1}{z}-z\right)\Gamma(z)\gamma(z),\end{aligned}$$

Since $\Gamma(z+2)=z(z+1)\Gamma(z)$ and $\gamma(z+2)=\gamma(z)/z(z+1)$.

Putting $u_1(z)=\Gamma(z)$ and $u_2(z)=\gamma(z)$ in (19) respectively, we see that the rational functions

$$f_1(z)=-\frac{z^4-2z^3-z^2+1}{(z^2-z+1)(z^2-z-1)}$$

and

$$f_2(z)=\frac{z^4-2z^3-z^2+4z-1}{(z^2-z+1)(z^2-z-1)}$$

satisfy the difference Riccati equation (4) with

$$A(z)=-\frac{z^8-6z^6+15z^4-2z^2+1}{(z^2+z+1)(z^2+z-1)(z^2-z+1)(z^2-z-1)}.$$

General solutions $f(z)$ to (4) with $A(z)$ above can be written as

$$f(z)=1+R(z)\frac{Q_1(z)z^2\Gamma(z)+Q_2(z)}{z(Q_1(z)\Gamma(z)+Q_2(z))},$$

with

$$R(z)=\frac{-2z^2(z-2)}{(z^2-z+1)(z^2-z-1)},$$

where $Q_j(z)$, $j=1,2$ are periodic functions of period 1. This shows that the difference Riccati equation (4) possesses infinitely many transcendental meromorphic solutions and two distinct rational solutions $f_1(z)$ and $f_2(z)$. By means of Proposition 2.1 in [12], we see that there is no rational solution other than $f_1(z)$ and $f_2(z)$ in this case.

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