木村 直文

Study on the solutions of Fermat type functional equation

(フェルマー型函数方程式の解に関する研究)

放送大学大学院文化科学研究科文化科学専攻

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Study on the solutions of Fermat type functional equation

by

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Abstract

This dissertation summarizes the research results on the solution of Fermat type functional equations, $(*)\ f_1^n + f_2^n + \cdots + f_k^n = 1$ where n and k are positive integers, see [17], [10]. Our focus is on equations of the form $(*)$ where it is not known whether there exist non-constant solutions in one or more of the following four classes of functions: meromorphic functions, rational functions, entire functions, polynomials.

To explain the contents, in Chapter 2, we summarize the mathematical tools used in this dissertation. It also includes properties for rational functions.

In Chapter 3, the Fermat type functional equations $(**)\ f^n + g^n + h^n = 1$ are considered in the complex plane. Alternative proofs for the known results for entire and meromorphic solutions to $(**)$ are given. Moreover, some conditions on degrees of polynomial solutions are given.

In Chapter 4, the Fermat type functional equations $(*)$ are considered in the complex plane. For such equations, we obtain estimates on Nevanlinna functions that transcendental solutions of $(*)$ would have to satisfy, as well as analogous estimates for non-constant rational solutions. As an application, it is shown that transcendental entire solutions of $(*)$ when $n = k(k-1)$ with $k \geq 3$, would have to satisfy a certain differential equation, which is a generalization of the known result when $k = 3$. Alternative proofs for the known non-existence theorems for entire and polynomial solutions of $(*)$ are given. Moreover, some restrictions on degrees of polynomial solutions are discussed.

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Preface

Human beings may be said to have lived with numbers. Replacing objects with the concept of numbers and counting is thought to have led to natural numbers, and the concept of numbers has expanded over time. Simultaneously, of course, the concept of numbers in problems has been broadened. The problem of Fermat type functional equations, which is the subject of this dissertation, can be considered as a problem involving such properties.

The problem we are working on in this dissertation is whether there exist meromorphic and entire functions f_1, f_2, $\ldots$, f_k which satisfy the Fermat type functional equation

$$f_1^n + f_2^n + \cdots + f_k^n = 1$$

where $n \geq k \geq 2$ are integers, see e.g. [9], [13].

Here, we would like to explain a little about the historical background related to this issue. Looking at the problem from the perspective of expanding the concept of numbers, as the name Fermat type functional equations suggest, the starting point of the problem is Fermat's Last Theorem of the famous French Pierre de Fermat. He wrote in the margin of the book Arithmetica by the ancient Greek mathematician Diophantus in the 1630s, "Equation $x^n + y^n = z^n$ $xyz \neq 0$ and $n > 2$. There is no set of natural numbers (x, y, z) that satisfies this equation. I have discovered a truly marvelous demonstration, which this margin is too narrow to contain." is later known as Fermat's Last Theorem. Fermat's Last Theorem has long been considered by mathematicians and math enthusiasts because of its ease of understanding, but it has been regarded as an unsolvable problem. This was fully proved by Andrew John Wiles of the United Kingdom in 1995. It has been 330 years since Fermat died.

The genealogy of Fermat's Problem is that in 1770, Edward Waring of the United Kingdom proposed Waring's problem, "Given a positive integer k, does the equation $n = x_1^k + x_2^k + \cdots + x_s^k$ hold for every integer n, where s depends on k but not on n? If so, what is the smallest value of s for a given k?" Waring's problem was positively proved by David Hilbert of Germany in 1909. Waring's problem with polynomial was also treated as similar to classical Waring's problem, see e.g. [20]. Also, in Gross's paper in 1966 [4], we can see the extension of the function of the

solution to the Waring's problem to meromorphic functions solution.
In 1985 and 2014, Hayman of the United Kingdom published the paper that arranged the format of the problem so far. Among them, Fermat type functional equations are mentioned, see e.g. [9], [13], [14].

Contents

Chapter 1

Introduction

Fermat's Last Theorem: For natural numbers n of 3 or more, there is no set of natural numbers (x, y, z) such that $x^n + y^n = z^n$. Despite the challenges of many, Fermat's Last Theorem could not be solved for a very long time, but was finally proved by Andrew John Wiles in 1995. The problem of the Fermat type functional equation, which is the problem in this dissertation, has a shape very similar to Fermat's Last Theorem. However, the difference is that the number of terms and the types of functions considered as solutions are widespread. What's more, this is the most important point, but it's also different in that there are still open questions left.

The Fermat type functional equation is described below. $(*)$ $f_1^n + f_2^n + \cdots + f_k^n = 1$, where n and k are positive integers. Our focus is on equations of the form $(*)$ where it is not known whether there exist non-constant solutions in one or more of the following four classes of functions: meromorphic functions, rational functions, entire functions, polynomials.

This dissertation summarizes the results of our research on the solutions of such Fermat type functional equation. Specifically, this dissertation is integrated by adding the necessary items with the main contents of the papers [17] and [10] submitted and published in "Computational Methods and Function Theory" and "Proceedings of the Edinburgh Mathematical Society". The topics in each chapter are explained below. In Chapter 2, we provide basic knowledge about the range of complex functions used in this dissertation, Nevanlinna theory and Wronskian. It also defines unique notations and properties for rational functions. In Chapter 3, the Fermat type functional equations $(**)$ $f^n + g^n + h^n = 1$ are considered in the complex plane. Alternative proofs for the known results for entire and meromorphic solutions to $(**)$ are given in Corollary 3.4 and Corollary 3.2, respectively. Moreover, some conditions on degrees of polynomial solutions are given in Proposition 3.1.

In Chapter 4, the Fermat type functional equations $(*)$ are considered in the complex plane. For such equations, we obtain estimates on Nevanlinna functions that transcendental solutions of $(*)$ would have to satisfy in Theorem 4.2, as well as analogous estimates for non-constant rational solutions in Theorem 4.1. As an application, it is shown that transcendental entire solutions of $(*)$ when $n = k(k-1)$ with $k \geq 3$, would have to satisfy a certain differential equation in Corollary 4.1, which is a generalization of the known result when $k = 3$. Alternative proofs for the known non-existence theorems for entire and polynomial solutions of $(*)$ are given in Corollary 4.1 and Corollary 4.2, respectively. Moreover, some restrictions on degrees of polynomial solutions are discussed in Lemma 4.3 and Proposition 4.1. We give an example of the case $k = 3$, $n = 3$ and $\deg f = \deg g = \deg h = 3$ of polynomial solutions of (*).

Chapter 2

Preliminaries

In preparation for reading this dissertation, we explain an overview of the relevant parts of complex functions, Nevalinna theory, Wronskian and properties for rational functions. In summarizing this section, we referred to references [1], [26] for complex functions, references [12], [21], [23] for Nevanlinna theory and reference [21] for Wronskian. For more details, check the references.

2.1 Complex Functions

Analytic Functions

If the derivative $f'(z)$ exists at all points z of a domain D, then $f(z)$ is said to be analytic in D and is referred to as an analytic function in D or a function analytic in D. The terms regular and holomorphic are sometimes used as synonyms for analytic.

A function $f(z)$ is said to be analytic at a point z_0 if there exists a neighborhood $|z - z_0| < \delta$, at all points of which $f'(z)$ exists.

Laurent Series

Suppose that $f(z)$ is single-valued and analytic in the ring-shaped region $E = \{r_1 < |z-a| < r_2\}$. Let C be an arbitrary circle in E centred at a. We can expand

$f(z)$ as

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots$$
$$= \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n} = \sum_{n=-\infty}^{\infty} a_n(z-a)^n,$$

where
$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-a)^{n+1}} d\zeta \qquad n = 0, \pm 1, \pm 2, \cdots .$$

Pole and Zero

A point at which $f(z)$ fails to be analytic is called a singular point or singularity of $f(z)$. Various types of singularities exist.

Isolated Singularities. The point $z = z_0$ is called an isolated singularity or isolated singular point of $f(z)$, if we can find $\delta > 0$ such that the circle $|z - z_0| = \delta$ encloses no singular point other than z_0 (i.e. there exists a deleted δ neighborhood of z_0 containing no singularity). If no such δ can be found, we call z_0 a non-isolated singularity.

If z_0 is not a singular point and we can find $\delta > 0$ such that $|z - z_0| = \delta$ encloses no singular point, then we call z_0 an ordinary point of $f(z)$.

Pole. If z_0 is an isolated singularity and we can find a positive integer n such that $\lim_{z\to z_0}(z - z_0)^n f(z) = A \neq 0$, then $z = z_0$ is called a pole of order n. If $n = 1$, z_0 is called a simple pole.

Zero. If $g(z) = (z - z_0)^n f(z)$, where $f(z_0) \neq 0$ and n is a positive integer, then $z = z_0$ is called a zero of order n of $g(z)$. If $n = 1$, z_0 is called a simple zero. In such a case, z_0 is a pole of order n of the function $1/g(z)$.

Polynomials

Polynomials are defined by

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

where $a_0 \neq 0, a_1, \cdots, a_n$, are complex constants and n is a positive integer called the degree of the polynomial $P(z)$.

Rational Functions

Rational functions are defined by

$$w = \frac{P(z)}{Q(z)}$$

where $P(z)$ and $Q(z)$ are polynomials. This function is analytical except for the point $Q(z) = 0$. Here, it is assumed that the common factor of $P(z)$ and $Q(z)$ has already been eliminated.

Entire Functions

A function that is analytic everywhere in the finite plane (i.e. everywhere except at ∞) is called an entire function. The functions e^z, $\sin z$, $\cos z$ are entire functions for example.

Meromorphic Functions

A function that is analytic everywhere in the finite plane except at a finite number of poles is called a meromorphic function. For example, $z/((z-1)(z+3)^2)$, which is analytic everywhere in the finite plane except at the poles $z = 1$ (simple pole) and $z = -3$ (pole of order 2), is a meromorphic function, and Γ function is analytic except at the poles $0, -1, -2, \cdots$.

In this dissertation, 'meromorphic' means meromorphic in the whole complex plan $\mathbb{C}$, and we consider meromorphic functions by dividing them into flowing four sets, depending on whether there is a pole or not, and whether it is transcendental.

M: Set of transcendental meromorphic functions having at least one pole
R: Set of rational functions having at least one pole
E: Set of transcendental entire functions
P: Set of polynomials

2.2 Nevanlinna Theory

In this dissertation, we use standard notations in the Nevanlinna theory. Therefore, the related items are summarized below, see e.g. [12], [21], [23].

Definition 2.1 (**Unintegrated counting function**)

Let f be a meromorphic function, not being identically equal to $a \in \mathbb{C}$. Let $i(z,a,f)$ denote the multiplicity of an a-point of f at z. Then we define

$$n(r,a,f) = n\left(r, \frac{1}{f-a}\right) = n(r,a) = \sum_{\substack{|z|\le r\\ f(z)=a}} i(z,a,f),$$

i.e. $n(r,a,f)$ counts the number of the roots of $f(z)=a$ in $|z| \le r$, each root according to its multiplicity. For the poles of f, we define similarly

$$n(r,\infty,f) = n(r,f) = n(r,\infty) = \sum_{\substack{|z|\le r\\ f(z)=\infty}} i(z,\infty,f).$$

Another method of counting the number of poles is to count all the multiplicities as 1, or to count only the poles having an order of 2 or more. The number of such counting methods is as follows

$$\overline{n}(r,\infty,f) = \overline{n}(r,f) = \overline{n}(r,\infty) = \sum_{\substack{|z|\le r\\ f(z)=\infty}} 1,$$

$$\begin{aligned} n_1(r,\infty,f) &= n_1(r,f) = n_1(r,\infty),\\ &= n(r,f) - \overline{n}(r,f) = \sum_{\substack{|z|\le r\\ f(z)=\infty}} \big(i(z,\infty,f)-1\big). \end{aligned}$$

Definition 2.2 (**Counting function**)

For a meromorphic function f, we define

$$N(r,a,f) = N\left(r,\frac{1}{f-a}\right) = \int_0^r \frac{n(t,a,f)-n(0,a,f)}{t}\,dt + n(0,a,f)\log r$$

supposing $f \not\equiv a \in \mathbb{C}$, and

$$N(r,f) = \int_0^r \frac{n(t,f)-n(0,f)}{t}\,dt + n(0,f)\log r.$$

Further, corresponding to other methods of counting poles

$$\overline{N}(r,f) = \int_0^r \frac{\overline{n}(t,f)-\overline{n}(0,f)}{t}\,dt + \overline{n}(0,f)\log r,$$

$$N_1(r,f) = \int_0^r \frac{n_1(t,f)-n_1(0,f)}{t}\,dt + n_1(0,f)\log r$$

and

$$N_1(r,f) = N(r,f) - \overline{N}(r,f).$$

Definition 2.3 (**Proximity function**)
For any real number $\alpha > 0$, we define

$$\log^+ \alpha = \max\{\log \alpha, 0\}.$$

For a meromorphic function f, we define

$$m(r, a, f) = m\left(r, \frac{1}{f-a}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{1}{f(re^{i\theta}) - a}\right| \, d\theta$$

supposing $f(z) \not\equiv a \in \mathbb{C}$, and

$$m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta.$$

Definition 2.4 (**Characteristic function**)
The characteristic function of a meromorphic function f is defined as

$$T(r, f) = m(r, f) + N(r, f).$$

The order $\sigma(f)$ of growth of a meromorphic function f is defined by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

By definitions, we may write

$$m(r, a, f) + N(r, a, f) = T\left(r, \frac{1}{f-a}\right).$$

Theorem 2.1 (**The first main theorem of Nevanlinna**)
Let f be a transcendental meromorphic function and let $a \in \mathbb{C}$. Then

$$T(r, f) = T\left(r, \frac{1}{f-a}\right) + O(1), \quad \text{as } r \to \infty.$$

Let f be a transcendental meromorphic function, we denote by $S(r, f)$ any quantity that satisfies $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set of r of finite linear measure. When a meromorphic function $a(z)$ satisfies $T(r, a) = S(r, f)$, $a(z)$ is said to be a small function with respect to $f(z)$.

Lemma 2.1 Let f be a transcendental meromorphic function. Then

$$m\left(r, \frac{f'}{f}\right) = S(r,f)$$

and if f is of finite order of growth, then

$$m\left(r, \frac{f'}{f}\right) = O(\log r).$$

Theorem 2.2 (**The second main theorem of Nevanlinna**)
Let f be a transcendental meromorphic function, let $q \geq 2$ and let $a_1, \ldots, a_q \in \mathbb{C}$ be distinct complex numbers. Then

$$m(r,f) + \sum_{j=1}^{q} m\left(r, \frac{1}{f-a_j}\right) + N_1(r,f) + N\left(r, \frac{1}{f'}\right) \leq 2T(r,f) + S(r,f).$$

Proposition 2.1 *Let $f, f_1, \cdots, f_n$ be meromorphic functions and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $\alpha\delta - \beta\gamma \neq 0$. Then*

(a) $T(r, f_1 \cdots f_n) \leq \sum_{i=1}^{n} T(r, f_i)$ *for* $r \geq 1$,

(b) $T(r, f^n) = nT(r,f),\ n \in \mathbb{N}$,

(c) $T\left(r, \sum_{i=1}^{n} f_i\right) \leq \sum_{i=1}^{n} T(r, f_i) + \log n$ *for* $r \geq 1$,

(d) $T\left(r, \dfrac{\alpha f + \beta}{\gamma f + \delta}\right) = T(r,f) + O(1)$,

assuming $f(z) \not\equiv -\delta/\gamma$.

2.3 Wronskian

Definition 2.5 (**Wronskian**)
The Wronskian $W(f_1, f_2, \cdots, f_n)$ of the meromorphic functions $f_1, f_2, \cdots, f_n$ is given by

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

Moreover, we denote, for $\nu = 0, \cdots, n-1$, by

$$W_\nu(f_1, f_2, \cdots, f_n)$$

the determinant which comes from $W(f_1, f_2, \cdots, f_n)$ by replacing the row $(f_1^{(\nu)}, f_2^{(\nu)}, \cdots, f_n^{(\nu)})$ by $(f_1^{(n)}, f_2^{(n)}, \cdots, f_n^{(n)})$.

Proposition 2.2 *Let $f_1, f_2, \cdots, f_n$ be meromorphic functions. Then $W(f_1, f_2, \cdots, f_n)$ vanishes identically if and only if $f_1, f_2, \cdots, f_n$ are linearly dependent over $\mathbb{C}$.*

Proposition 2.3 *Let $f_1, f_2, \cdots, f_n$ be meromorphic functions and $c_1, c_2, \cdots, c_n$ be complex numbers. Then*

(a) $W(c_1 f_1, c_2 f_2, \cdots, c_n f_n) = c_1 c_2 \cdots c_n W(f_1, f_2, \cdots, f_n)$.

(b) $W\left(1, z, \cdots, \dfrac{z^{n-1}}{(n-1)!}, g\right) = g^{(n)}$.

(c) $W(f_1, f_2, \cdots, f_n, 1) = (-1)^n W(f_1', f_2', \cdots, f_n')$.

(d) $W(gf_1, gf_2, \cdots, gf_n) = g^n W(f_1, f_2, \cdots, f_n)$.

(e) $W(f_1, f_2, \cdots, f_n) = f_1^n W\left(\left(\dfrac{f_2}{f_1}\right)', \cdots, \left(\dfrac{f_n}{f_1}\right)'\right)$.

2.4 Properties for Rational Functions

We prepare some notations for rational functions, and give a lemma in the remaining part of this chapter. Let R be a rational function. Let $\mathfrak{n}(R)$ denote the number poles of R in $\mathbb{C}$, where each pole is counted the same number of times as its multiplicity. Write $R = R_N/R_D$, where R_N and R_D are relatively prime polynomials. Obviously, we have $\mathfrak{n}(R) = \deg R_D$. Define

$$\mathfrak{m}(R) = \max(\deg R_N - \deg R_D,\ 0). \tag{2.1}$$

Using $\mathfrak{n}(R) = \deg R_D$, we see that

$$\deg R = \mathfrak{m}(R) + \mathfrak{n}(R) \tag{2.2}$$

and for any $a \in \mathbb{C}$

$$\deg R = \deg\left(\frac{1}{R-a}\right) = \mathfrak{m}\left(\frac{1}{R-a}\right) + \mathfrak{n}\left(\frac{1}{R-a}\right). \tag{2.3}$$

Concerning the properties of $\mathfrak{m}(R)$, we mention the following lemma.

Lemma 2.2 *Let P be a polynomial, and let R and Q be rational functions. Then*

(i) *For any non-negative integers $j > k$, either $\deg P \le k$ or it holds*

$$\mathfrak{m}\left(\frac{P^{(j)}}{P^{(k)}}\right) = 0. \tag{2.4}$$

(ii) *We have*

$$\mathfrak{m}(RQ) \le \mathfrak{m}(R)+\mathfrak{m}(Q) \quad \textit{and} \quad \mathfrak{m}(R+Q) \le \max\left(\mathfrak{m}(R),\ \mathfrak{m}(Q)\right). \tag{2.5}$$

Proof of Lemma 2.2 (i) Since $\deg P^{(j)} < \deg P^{(k)}$ when $\deg P > k$, by definition we obtain (2.4).

(ii) We write $R = R_N/R_D$ with relatively prime polynomials R_N and R_D, and $Q = Q_N/Q_D$ with relatively prime polynomials Q_N and Q_D. We have

$$\begin{aligned}\mathfrak{m}(RQ) = \mathfrak{m}\left(\frac{R_N Q_N}{R_D Q_D}\right) &= \max\left((\deg R_N + \deg Q_N) - (\deg R_D + \deg Q_D),\ 0\right)\\ &\le \max(\deg R_N - \deg R_D,\ 0) + \max(\deg Q_N - \deg Q_D,\ 0)\\ &= \mathfrak{m}(R) + \mathfrak{m}(Q),\end{aligned}$$

which shows the first inequality in (2.5). For the second inequality in (2.5), we estimate

$$\begin{aligned}&\mathfrak{m}(R+Q) = \mathfrak{m}\left(\frac{R_N Q_D + R_D Q_N}{R_D Q_D}\right)\\ &\le \max\Big(\max(\deg R_N + \deg Q_D,\ \deg R_D + \deg Q_N) - (\deg R_D + \deg Q_D),\ 0\Big)\\ &\le \max\Big(\max(\deg R_N - \deg R_D,\ 0),\ \max(\deg Q_N - \deg Q_D,\ 0)\Big)\\ &= \max(\mathfrak{m}(R),\ \mathfrak{m}(Q)).\end{aligned}$$

We have thus proved Lemma 2.2. □

Remark 2.1 Let R be a rational function. We have $\mathfrak{m}(R^n) = n\mathfrak{m}(R)$ for all $n \in \mathbb{N}$ and $\mathfrak{m}(cR) = \mathfrak{m}(R)$ for all $c \in \mathbb{C}\backslash\{0\}$. These properties will be used in the proofs of Theorem 3.2 and Corollary 3.3 in Chapter 3 below.

Chapter 3

Case $f^n + g^n + h^n = 1$

3.1 Background of Case $f^n + g^n + h^n = 1$

In Chapter 3, we are concerned with the problem whether there exist non-constant entire and meromorphic functions f, g and h which satisfy the Fermat type functional equation

$$f^n + g^n + h^n = 1, \tag{3.1}$$

where $n \geq 2$ is an integer.

It has been determined for which positive integers n, the functional equation $f^n + g^n = 1$ has non-constant entire and meromorphic solutions f and g, see e.g. [2], [3], [7], [9], [14]. Hence, we assume in (3.1) that f^n, g^n and h^n are linearly independent, otherwise (3.1) reduces to $f^n + g^n = 1$.

We know that there exist solutions of non-constant rational functions to (3.1) when $n \leq 5$, and know that there do not exist non-constant rational functions satisfying (3.1) when $n \geq 8$.

For the cases $n \leq 3$, we know that there exist non-constant polynomial solutions to (3.1), and there do not exist non-constant polynomial solutions to (3.1) when $n \geq 6$.

To our best knowledge, the cases $n = 7$ and 6 for solutions of non-constant rational functions, and the cases $n = 5$ and 4 for non-constant polynomial solutions are still open, see e.g. [8], [9], [19].

Example 3.1 We recall examples for $n = 4$ and $n = 5$. Let $a(z) \not\equiv 0$ be a meromorphic function, where 'meromorphic' means meromorphic in the whole complex plane $\mathbb{C}$. The following functions

$$f(z) = \frac{1}{\sqrt[4]{8}}\left(a(z)^3 + \frac{1}{a(z)}\right), \; g(z) = \frac{1}{\sqrt[4]{-8}}\left(a(z)^3 - \frac{1}{a(z)}\right),$$
$$h(z) = \sqrt[4]{-1}\, a(z)^2 \tag{3.2}$$

satisfy (3.1) for $n = 4$, and functions

$$f(z) = \frac{1}{3}\left((2-\sqrt{6})a(z) + 1 + \frac{2+\sqrt{6}}{a(z)}\right), \tag{3.3}$$

$$g(z) = \frac{1}{6}\Bigg(\Big((\sqrt{6}-2) + (3\sqrt{2}-2\sqrt{3})i\Big)\, a(z) + 2 - \frac{(\sqrt{6}+2) - (3\sqrt{2}+2\sqrt{3})i}{a(z)}\Bigg), \tag{3.4}$$

$$h(z) = \frac{1}{6}\Bigg(\Big((\sqrt{6}-2) - (3\sqrt{2}-2\sqrt{3})i\Big)\, a(z) + 2 - \frac{(\sqrt{6}+2) + (3\sqrt{2}+2\sqrt{3})i}{a(z)}\Bigg) \tag{3.5}$$

satisfy (3.1) for $n = 5$, see [6], [7], [9], [11].

Example 3.2 Regarding the case when $n = 6$ in (3.1), we have examples three transcendental meromorphic functions f, g and h.
In [5], Gundersen constructed the example for the case $n = 6$ by the following algebraic identity, and by Rellich's result on elliptic functions, see e.g. [15], [25], [28].

Lemma 3.1 If a and b are any two constants, then

$$(a-b)^6 + (a+b)^6 = (1 + \frac{11}{2}i)(a^2 + (\frac{3}{5} - \frac{4}{5}i)b^2)^3 + (1 - \frac{11}{2}i)(a^2 + (\frac{3}{5} + \frac{4}{5}i)b^2)^3. \tag{3.6}$$

Theorem 3.1 *Let $Q_3(z)$ be a polynomial of degree three that has three distinct zeros. Then every non-constant solution F of the differential equation*

$$(F')^2 = Q_3(F) \tag{3.7}$$

is an elliptic function.

Setting a rational function as a in Example 3.1, we ascertain that there exist solutions of non-constant rational functions to (3.1) when $n = 4$ and 5.

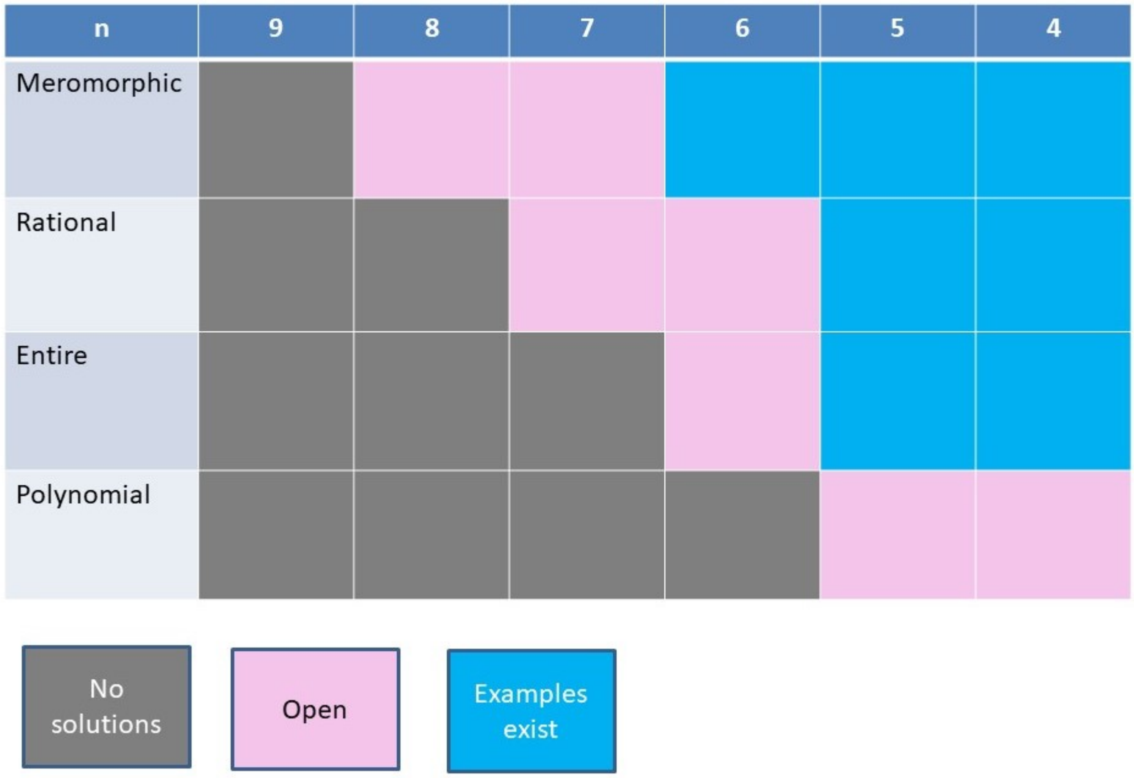
Concerning transcendental cases for (3.1), there exist transcendental meromorphic solutions to (3.1) when $n \leq 6$, see [5], and know that there is no transcendental meromorphic solutions to (3.1) when $n \geq 9$, see [9], [13]. For the case $n \leq 5$, we know that there exist transcendental entire solutions to (3.1), and there is no transcendental entire solutions to (3.1) when $n \geq 7$. Setting an entire function of the form e^α with an entire function α as a in Example 3.1, we ascertain that there exist transcendental entire solutions to (3.1) when $n = 4$ and 5. The cases $n = 8$ and 7 for transcendental meromorphic solutions, and the case $n = 6$ for transcendental entire solutions are still open, see e.g. [8], [19].

In [16], Ishizaki investigated transcendental solutions to (3.1) and obtained alternative proofs of the known results for transcendental meromorphic and entire solutions. Ishizaki also obtained that transcendental meromorphic solutions to (3.1) for $n = 8$ and transcendental entire solutions to (3.1) for $n = 6$ satisfy some differential equations if such solutions exist.

We recall the differential equations that solutions of (3.1) satisfy, and we introduce auxiliary functions in Section 3.2. Preliminary lemmas are mentioned in Section 3.3, and we give their proofs. Section 3.4 devotes to estimates on poles and zeros of solutions to (3.1). We are concerned with meromorphic solutions to (3.1) in Section 3.5 and with entire solutions to (3.1) in Section 3.6, and we state the main theorems. We apply them to (3.1) in order to give alternative proofs of the known results in both Sections 3.5 and 3.6, respectively. In Section 3.7, we give conditions on degrees of polynomial solutions for $n \leq 5$ if such solutions exist.

We summarize the situation for $k = 3$ in Figure 3.1.

Fig. 3.1: Known Results for $f^n + g^n + h^n = 1$



3.2 Differential Equations

For functions f_1, f_2, $\ldots$, f_n, $n \geq 2$, we denote by $W(f_1, f_2, \ldots, f_n)$ the Wronskian of f_1, $f_2, \ldots,$ f_n. We assume that there exist functions f, g and h which satisfy the functional equation (3.1). For the sake of simplicity, we put $f^n = F$, $g^n = G$ and $h^n = H$, and define

$$\Delta = \frac{W(F, G, H)}{FGH} = \begin{vmatrix} 1 & 1 & 1 \\ \dfrac{F'}{F} & \dfrac{G'}{G} & \dfrac{H'}{H} \\ \dfrac{F''}{F} & \dfrac{G''}{G} & \dfrac{H''}{H} \end{vmatrix}. \tag{3.8}$$

Using (3.1), we have

$$\begin{aligned}
\Delta &= \begin{vmatrix} \dfrac{G'}{G} & \dfrac{H'}{H} \\ \dfrac{G''}{G} & \dfrac{H''}{H} \end{vmatrix} + \begin{vmatrix} \dfrac{H'}{H} & \dfrac{F'}{F} \\ \dfrac{H''}{H} & \dfrac{F''}{F} \end{vmatrix} + \begin{vmatrix} \dfrac{F'}{F} & \dfrac{G'}{G} \\ \dfrac{F''}{F} & \dfrac{G''}{G} \end{vmatrix} \\
&= \frac{(FG'H'' + F'G''H + F''GH') - (FG''H' + F'GH'' + F''G'H)}{FGH} \\
&= \frac{1}{FGH}\Big(FG'(-F'' - G'') + F'G''(1 - F - G) + F''G(-F' - G') \\
&\qquad - \big(FG''(-F' - G') + F'G(-F'' - G'') + F''G'(1 - F - G)\big)\Big) \\
&= \frac{1}{H}\left(\frac{F'G''}{FG} - \frac{F''G'}{FG}\right) \\
&= \frac{1}{H}\begin{vmatrix} \dfrac{F'}{F} & \dfrac{F''}{F} \\ \dfrac{G'}{G} & \dfrac{G''}{G} \end{vmatrix} \\
&= \frac{W(F', G')}{FGH}.
\end{aligned}$$

Then

$$\Delta = \frac{W(F',G')}{FGH} = \frac{1}{H}\begin{vmatrix} \frac{F'}{F} & \frac{G'}{G} \\ \frac{F''}{F} & \frac{G''}{G} \end{vmatrix} = \frac{\delta_{FG}}{H} \tag{3.9}$$

$$= \frac{W(G',H')}{FGH} = \frac{1}{F}\begin{vmatrix} \frac{G'}{G} & \frac{H'}{H} \\ \frac{G''}{G} & \frac{H''}{H} \end{vmatrix} = \frac{\delta_{GH}}{F} \tag{3.10}$$

$$= \frac{W(H',F')}{FGH} = \frac{1}{G}\begin{vmatrix} \frac{H'}{H} & \frac{F'}{F} \\ \frac{H''}{H} & \frac{F''}{F} \end{vmatrix} = \frac{\delta_{HF}}{G}. \tag{3.11}$$

It follows from (3.8) and (3.9),

$$\begin{aligned} W(F,G,H) &= W(F',G') \\ &= \begin{vmatrix} F' & G' \\ F'' & G'' \end{vmatrix} \\ &= F'G'' - F''G' \\ &= nf'f^{n-1}n\Big(g''g^{n-1} + (n-1)g'g'g^{n-2}\Big) - n\Big(f''f^{n-1} + (n-1)f'f'f^{n-2}\Big)ng'g^{n-1} \\ &= n^2 f^{n-2}g^{n-2}\Big((n-1)f'g'(fg' - f'g) + fg(f'g'' - f''g')\Big) \\ &= n^2 f^{n-2}g^{n-2}\Big((n-1)f'g'W(f,g) + fgW(f',g')\Big) \end{aligned}$$

Hence

$$\begin{aligned} W(F,G,H) &= W(f^n,g^n,h^n) \\ &= n^2 f^{n-2}g^{n-2}V_1 = n^2 g^{n-2}h^{n-2}V_2 = n^2 h^{n-2}f^{n-2}V_3, \end{aligned} \tag{3.12}$$

where

$$V_1 = (n-1)f'g'W(f,g) + fgW(f',g'), \tag{3.13}$$
$$V_2 = (n-1)g'h'W(g,h) + ghW(g',h'), \tag{3.14}$$
$$V_3 = (n-1)h'f'W(h,f) + hfW(h',f'). \tag{3.15}$$

We write

$$W(F,G,H) = W(f^n,g^n,h^n) = n^2 f^{n-2}g^{n-2}h^{n-2}V. \tag{3.16}$$

Then we have,

$$V_1 = h^{n-2}V, \quad V_2 = f^{n-2}V, \quad V_3 = g^{n-2}V, \tag{3.17}$$

and we write

$$\begin{aligned}
V = \frac{V_1}{h^{n-2}} &= \frac{1}{h^{n-2}}\left((n-1)\, f'g'W(f,g) + fgW(f',g')\right) \\
&= \frac{f^2g^2}{h^{n-2}} \cdot \frac{f'g'}{fg}\left((n-1)\frac{1}{fg}(fg' - f'g) + \frac{1}{f'g'}(f'g'' - f''g')\right) \\
&= \frac{f^2g^2}{h^{n-2}}\frac{f'g'}{fg}\left((n-1)\left(\frac{g'}{g} - \frac{f'}{f}\right) + \left(\frac{g''}{g'} - \frac{f''}{f'}\right)\right) \\
&= \frac{f^2g^2}{h^{n-2}} \cdot \frac{f'g'}{fg}\left((n-1)\begin{vmatrix} 1 & 1 \\ \frac{f'}{f} & \frac{g'}{g} \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ \frac{f''}{f'} & \frac{g''}{g'} \end{vmatrix}\right) = \frac{f^2g^2}{h^{n-2}}\,\eta_{fg}
\end{aligned} \tag{3.18}$$

Similarly,

$$V = \frac{V_2}{f^{n-2}} = \frac{g^2h^2}{f^{n-2}} \cdot \frac{g'h'}{gh}\left((n-1)\begin{vmatrix} 1 & 1 \\ \frac{g'}{g} & \frac{h'}{h} \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ \frac{g''}{g'} & \frac{h''}{h'} \end{vmatrix}\right) = \frac{g^2h^2}{f^{n-2}}\,\eta_{gh} \tag{3.19}$$

$$V = \frac{V_3}{g^{n-2}} = \frac{h^2f^2}{g^{n-2}} \cdot \frac{h'f'}{hf}\left((n-1)\begin{vmatrix} 1 & 1 \\ \frac{h'}{h} & \frac{f'}{f} \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ \frac{h''}{h'} & \frac{f''}{f'} \end{vmatrix}\right) = \frac{h^2f^2}{g^{n-2}}\,\eta_{hf}. \tag{3.20}$$

By (3.8) and (3.16), we have

$$n^2V = f^2g^2h^2\Delta. \tag{3.21}$$

Furthermore, it follows from (3.18) to (3.20),

$$V^3 = \frac{\eta_{fg}\eta_{gh}\eta_{hf}}{f^{n-6}g^{n-6}h^{n-6}}. \tag{3.22}$$

When f, g and h are polynomials, a polynomial $W(F,G,H)$ has factors $f^{n-2}g^{n-2}$, $g^{n-2}h^{n-2}$ and $h^{n-2}f^{n-2}$ by (3.12). This implies that $W(F,G,H)$ has a factor $f^{n-2}g^{n-2}h^{n-2}$, and hence V is a polynomial. Considering the case f, g and h are transcendental entire functions, we similarly obtain that V is an entire function. In particular, V reduces to a small function with respect to f, g and h when $n = 6$ if such solutions exist, see [16, Proposition 6.1]. It is known that there exist transcendental entire solutions to (3.1) when $n \leq 5$. We explicitly compute V of the solutions mentioned in Example 3.1.

Example 3.3 For the solutions f, g and h to (3.1) given by (3.2), we have

$$V = -24a(a')^3. \tag{3.23}$$

This implies that V is not a small function in general with respect to f, g and h in this case. Consider the solutions f, g and h to (3.1) given by (3.3) to (3.5). We set $a = e^{\alpha}$ with an entire function α. Then we ascertain that f, g and h are transcendental entire solutions. We have

$$V = \frac{4i}{\sqrt{3}}\left(\frac{a'}{a}\right)^3 = \frac{4i}{\sqrt{3}}\left(\alpha'\right)^3, \tag{3.24}$$

which is a small function with respect to f, g and h in this case.

In general, V is a rational functions function when f, g and h are solutions of rational functions to (3.1) if such solutions exist, and V is a meromorphic functions function when f, g and h are transcendental meromorphic solutions to (3.1) if such solutions exist. In particular, V reduces to a small function with respect to f, g and h when $n = 8$, see [16, Proposition 5.1]. We give the definition of a small function with respect to f, g and h later in Subsection 3.5.2.

3.3 Preliminary Lemmas

We first recall certain elementary properties of the Wronskian. Let $u \not\equiv 0$ and $v \not\equiv 0$ be meromorphic functions. Define

$$\delta(z) = \frac{W(u'(z), v'(z))}{u(z)v(z)} = \begin{vmatrix} \dfrac{u'(z)}{u(z)} & \dfrac{v'(z)}{v(z)} \\ \dfrac{u''(z)}{u(z)} & \dfrac{v''(z)}{v(z)} \end{vmatrix}.$$

Suppose that u has a zero at z_0 of multiplicity ν. Then we may write in a neighbourhood of z_0 by the Laurent expansions.

$$\begin{aligned} u(z) &= a_\nu(z-z_0)^\nu + a_{\nu+1}(z-z_0)^{\nu+1} + a_{\nu+2}(z-z_0)^{\nu+2} + \cdots \\ u'(z) &= \nu a_\nu(z-z_0)^{\nu-1} + (\nu+1)a_{\nu+1}(z-z_0)^{\nu} + (\nu+2)a_{\nu+2}(z-z_0)^{\nu+1} + \cdots \\ u''(z) &= \nu(\nu-1)a_\nu(z-z_0)^{\nu-2} + (\nu+1)\nu a_{\nu+1}(z-z_0)^{\nu-1} \\ &\qquad + (\nu+2)(\nu+1)a_{\nu+2}(z-z_0)^{\nu} + \cdots . \end{aligned}$$

Then we have

$$\begin{aligned} \frac{u'(z)}{u(z)} =& \frac{\nu a_\nu(z-z_0)^{\nu-1} + (\nu+1)a_{\nu+1}(z-z_0)^{\nu} + (\nu+2)a_{\nu+2}(z-z_0)^{\nu+1} + \cdots}{a_\nu(z-z_0)^\nu + a_{\nu+1}(z-z_0)^{\nu+1} + a_{\nu+2}(z-z_0)^{\nu+2} + \cdots} \\ =& \frac{\nu a_\nu(z-z_0)^{\nu-1}\Big(1 + \dfrac{\nu+1}{\nu}\dfrac{a_{\nu+1}}{a_\nu}(z-z_0) + \dfrac{\nu+2}{\nu}\dfrac{a_{\nu+2}}{a_\nu}(z-z_0)^2 + \cdots\Big)}{a_\nu(z-z_0)^\nu\Big(1 + \dfrac{a_{\nu+1}}{a_\nu}(z-z_0) + \dfrac{a_{\nu+2}}{a_\nu}(z-z_0)^2 + \cdots\Big)} \\ =& \frac{\nu}{(z-z_0)} \cdot \frac{1}{1 + \dfrac{a_{\nu+1}}{a_\nu}(z-z_0) + \dfrac{a_{\nu+2}}{a_\nu}(z-z_0)^2 + \cdots} \\ &\cdot \Bigg(\Big(1 + \frac{a_{\nu+1}}{a_\nu}(z-z_0) + \frac{a_{\nu+2}}{a_\nu}(z-z_0)^2 + \cdots\Big) \\ &\qquad + \Big(\frac{1}{\nu}\frac{a_{\nu+1}}{a_\nu}(z-z_0) + \frac{2}{\nu}\frac{a_{\nu+2}}{a_\nu}(z-z_0)^2 + \cdots\Big)\Bigg) \\ =& \frac{\nu}{(z-z_0)} + \cdots . \end{aligned}$$

Similarly,

$$\frac{u''(z)}{u(z)} = \frac{\nu(\nu-1)}{(z-z_0)^2} + \cdots .$$

Suppose that u has a pole at z_0 of multiplicity μ. Then we may write in a neighbourhood of z_0 by the Laurent expansions.

$$\begin{aligned}
u(z) &= \frac{c_\mu}{(z-z_0)^\mu} + \frac{c_{\mu-1}}{(z-z_0)^{\mu-1}} + \frac{c_{\mu-2}}{(z-z_0)^{\mu-2}} + \cdots \\
u'(z) &= \frac{-\mu c_\mu}{(z-z_0)^{\mu+1}} + \frac{-(\mu-1)c_{\mu-1}}{(z-z_0)^{\mu}} + \frac{-(\mu-2)c_{\mu-2}}{(z-z_0)^{\mu-1}} + \cdots \\
u''(z) &= \frac{\mu(\mu+1)c_\mu}{(z-z_0)^{\mu+2}} + \frac{(\mu-1)\mu c_{\mu-1}}{(z-z_0)^{\mu+1}} + \frac{(\mu-2)(\mu-1)c_{\mu-2}}{(z-z_0)^{\mu}} + \cdots .
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{u'(z)}{u(z)} =& \frac{\dfrac{-\mu c_\mu}{(z-z_0)^{\mu+1}} + \dfrac{-(\mu-1)c_{\mu-1}}{(z-z_0)^{\mu}} + \dfrac{-(\mu-2)c_{\mu-2}}{(z-z_0)^{\mu-1}} + \cdots}{\dfrac{c_\mu}{(z-z_0)^\mu} + \dfrac{c_{\mu-1}}{(z-z_0)^{\mu-1}} + \dfrac{c_{\mu-2}}{(z-z_0)^{\mu-2}} + \cdots} \\
=& \frac{\dfrac{-\mu c_\mu}{(z-z_0)^{\mu+1}}\left(1 + \dfrac{\mu-1}{\mu}\dfrac{c_{\mu-1}}{c_\mu}(z-z_0) + \dfrac{\mu-2}{\mu}\dfrac{c_{\mu-2}}{c_\mu}(z-z_0)^2 + \cdots\right)}{\dfrac{c_\mu}{(z-z_0)^\mu}\left(1 + \dfrac{c_{\mu-1}}{c_\mu}(z-z_0) + \dfrac{c_{\mu-2}}{c_\mu}(z-z_0)^2 + \cdots\right)} \\
=& \frac{-\mu}{(z-z_0)} \cdot \frac{1}{1 + \dfrac{c_{\mu-1}}{c_\mu}(z-z_0) + \dfrac{c_{\mu-2}}{c_\mu}(z-z_0)^2 + \cdots} \\
& \cdot \Bigg(\left(1 + \frac{c_{\mu-1}}{c_\mu}(z-z_0) + \frac{c_{\mu-2}}{c_\mu}(z-z_0)^2 + \cdots\right) \\
& \qquad + \left(\frac{-1}{\mu}\frac{c_{\mu-1}}{c_\mu}(z-z_0) + \frac{-2}{\mu}\frac{c_{\mu-2}}{c_\mu}(z-z_0)^2 + \cdots\right)\Bigg) \\
=& \frac{-\mu}{(z-z_0)} + \cdots
\end{aligned}$$

Similarly,

$$\frac{u''(z)}{u(z)} = \frac{\mu(\mu+1)}{(z-z_0)^2} + \cdots .$$

Lemma 3.2

(i) *Suppose that z_0 is a pole of u, or a zero of u. If z_0 is neither a zero nor a pole of v, then δ is analytic at z_0 or has a pole at z_0 of multiplicity at most 2.*

(ii) *If z_0 is a pole of u, and z_0 is a zero of v of multiplicity at least 2, then δ has a pole at z_0 of multiplicity 3.*

(iii) *Suppose that* z_0 *is a pole of* u *of multiplicity* k_u*, and a pole of* v *of multiplicity* k_v*. If* $k_u = k_v$*, then* δ *is analytic at* z_0 *or has a pole at* z_0 *of multiplicity at most 2. If* $k_u \neq k_v$*, then* δ *has a pole at* z_0 *of multiplicity 3.*

(iv) *Suppose that* z_0 *is a zero of* u *of multiplicity* $m_u \geq 2$*, and a zero of* v *of multiplicity* $m_v \geq 2$*. If* $m_u = m_v$*, then* δ *is analytic at* z_0 *or has a pole at* z_0 *of multiplicity at most 2. If* $m_u \neq m_v$*, then* δ *has a pole at* z_0 *of multiplicity 3.*

3.4 Estimates for Poles and Zeros of f, g and h

Let w be a meromorphic function. If z_0 is a pole of multiplicity μ (≥ 1) for $w(z)$, then $\omega(z_0, w) = \mu$, and if $w(z_0) \neq \infty$, then $\omega(z_0, w) = 0$. The aim of this section is to obtain the following lemmas.

Lemma 3.3 *Suppose that* $n \geq 8$ *in* (3.1)*, and suppose there exist non-constant meromorphic functions* f*,* g *and* h *satisfying* (3.1)*. Let* V *be a meromorphic function given by* (3.16)*. Then the meromorphic function* V *does not have any poles.*

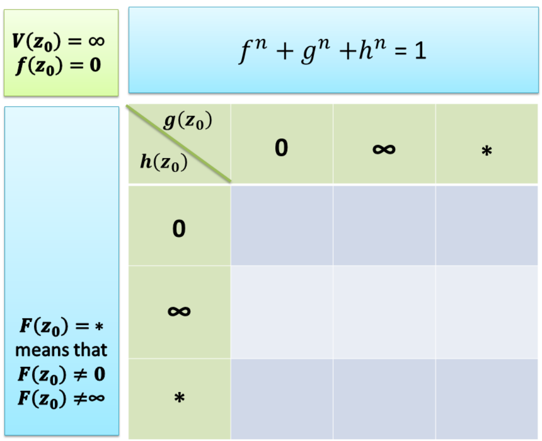
Lemma 3.4 *Suppose that* $n = 6$ *in* (3.1)*, and suppose there exist non-constant entire functions* f*,* g *and* h *satisfying* (3.1)*. Further, we suppose that* z_0 *is a multiple zero of at least one of* f*,* g *and* h*. Then* V *has a zeros at* z_0*, where* V *is the entire function given by* (3.16)*.*

Proof of Lemma 3.3 By means of (3.21), we see that z_0 is a zero or a pole of at least one of the f, g and h if we suppose that V has a pole at z_0.

We denote the sets of poles of f, g and h by $\mathcal{P}_f$, $\mathcal{P}_g$ and $\mathcal{P}_h$, respectively, and write $\mathcal{P} = \mathcal{P}_f \bigcup \mathcal{P}_g \bigcup \mathcal{P}_h$. Similarly, sets of zeros of f, g and h are denoted by $\mathcal{Z}_f$, $\mathcal{Z}_g$ and $\mathcal{Z}_h$, respectively, and we write $\mathcal{Z} = \mathcal{Z}_f \bigcup \mathcal{Z}_g \bigcup \mathcal{Z}_h$.

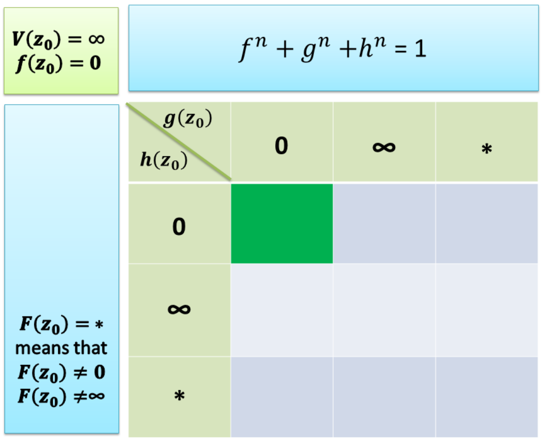
First we assert that any $z_0 \in \mathcal{Z}$ can not be a pole of V. We assume the contrary, namely, there exists $z_0 \in \mathcal{Z}$ such that $V(z_0) = \infty$. We may assume that $z_0 \in \mathcal{Z}_f$ without loss of generality. We use figures for explanation.

Fig. 3.2: Estimates for poles and zeros



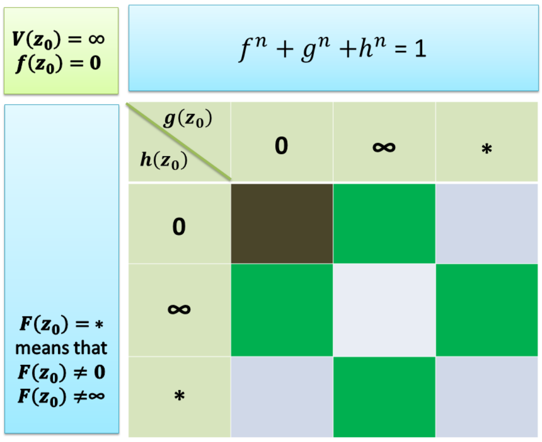
It is impossible that $z_0 \in \mathcal{Z}_g \bigcap \mathcal{Z}_h$ by (3.1).

Fig. 3.3: Estimates for poles and zeros



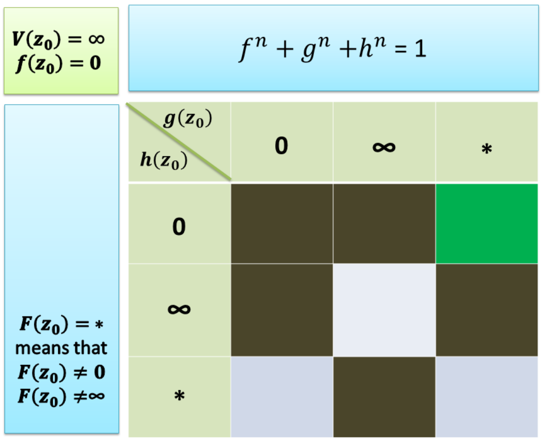
The case when only one of g and h has a pole at z_0 is impossible by (3.1), namely, the cases $z_0 \in \mathcal{P}_g \bigcap \mathcal{Z}_h$, $z_0 \in \mathcal{Z}_g \bigcap \mathcal{P}_h$, $z_0 \in (\mathbb{C} \setminus (\mathcal{Z}_g \bigcup \mathcal{P}_g)) \bigcap \mathcal{P}_h$ and $z_0 \in \mathcal{P}_g \bigcap (\mathbb{C} \setminus (\mathcal{Z}_h \bigcup \mathcal{P}_h))$ are impossible by (3.1).

Fig. 3.4: Estimates for poles and zeros



When $z_0 \in (\mathbb{C} \setminus (\mathcal{Z}_g \bigcup \mathcal{P}_g)) \bigcap \mathcal{Z}_h$, we have $V_3(z_0) = 0$ by (3.15), and hence $V(z_0) = 0$ by (3.20), a contradiction.

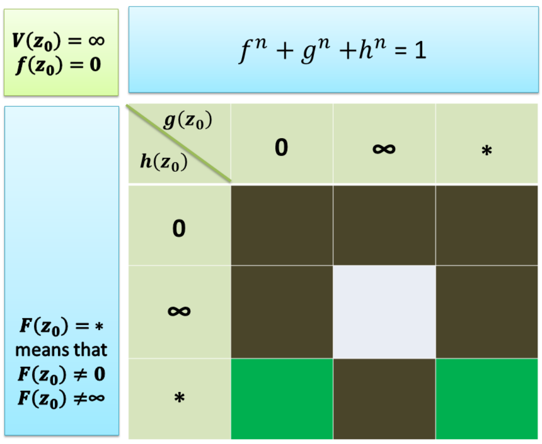
Fig. 3.5: Estimates for poles and zeros



Similarly, by (3.13) and (3.18), $V(z_0) = 0$ when $z_0 \in \mathcal{Z}_g \bigcap (\mathbb{C} \setminus (\mathcal{Z}_h \bigcup \mathcal{P}_h))$, a contradiction.

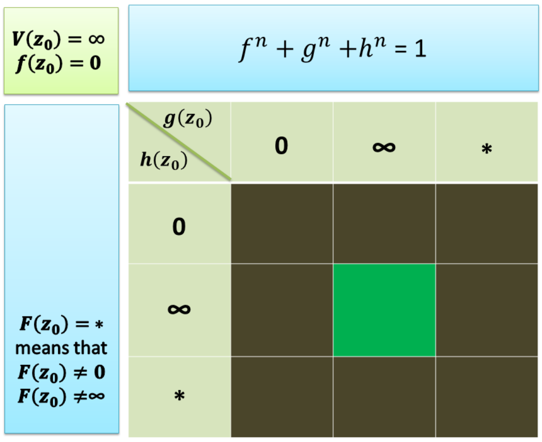
When $z_0 \in (\mathbb{C} \setminus (\mathcal{Z}_g \bigcup \mathcal{P}_g)) \bigcap (\mathbb{C} \setminus (\mathcal{Z}_h \bigcup \mathcal{P}_h))$, we have $V_3(z_0) \neq \infty$ by (3.15), and hence $V(z_0) \neq \infty$ by (3.20), a contradiction.

Fig. 3.6: Estimates for poles and zeros



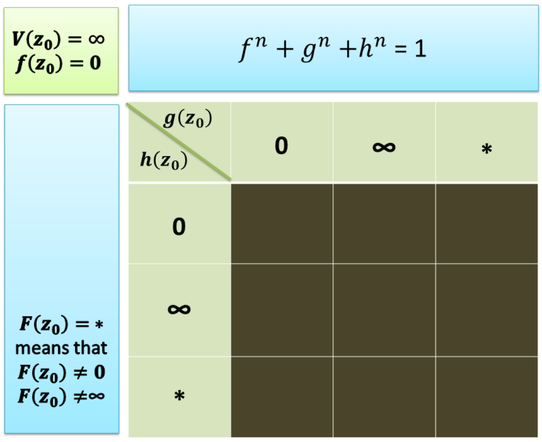
The case $z_0 \in \mathcal{P}_g \bigcap \mathcal{P}_h$ is left for consideration when $z_0 \in \mathcal{Z}_f$. By (3.18), we have $\omega(z_0, \eta_{fg}) \leq 3$, which implies that $\omega(z_0, f^2\eta_{fg}) \leq 1$. By (3.1), $\omega(z_0, g) = \omega(z_0, h)$, and hence g^2/h^{n-2} has a zero at z_0 of multiplicity $(n-4)\omega(z_0, g)$ since $n \geq 8$. Using (3.18), we conclude that V has a zero at z_0 of multiplicity at least $(n-4)\omega(z_0, g) - 1$, a contradiction.

Fig. 3.7: Estimates for poles and zeros



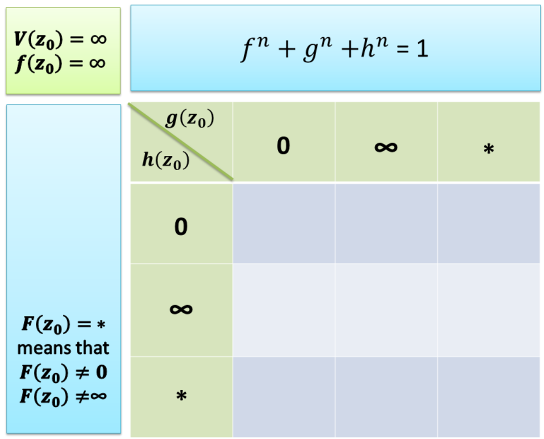
The first assertion follows.

Fig. 3.8: Estimates for poles and zeros



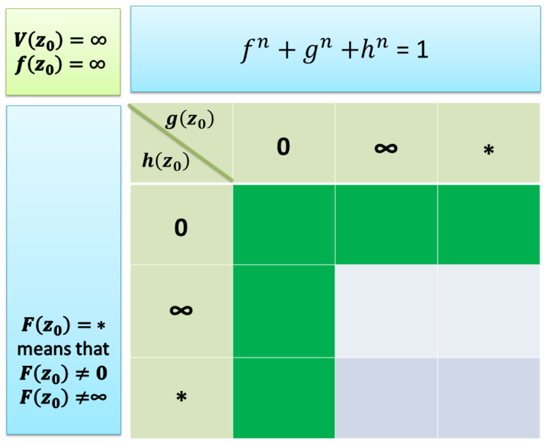
Secondly, we consider the case $z_0 \in \mathcal{P}$, and assert that if $z_0 \in \mathcal{P}$ is a pole of V, then $z_0 \in \mathcal{P}_f \bigcap \mathcal{P}_g \bigcap \mathcal{P}_h$. We may assume that $z_0 \in \mathcal{P}_f$ without loss of generality. We use figures for explanation.

Fig. 3.9: Estimates for poles and zeros



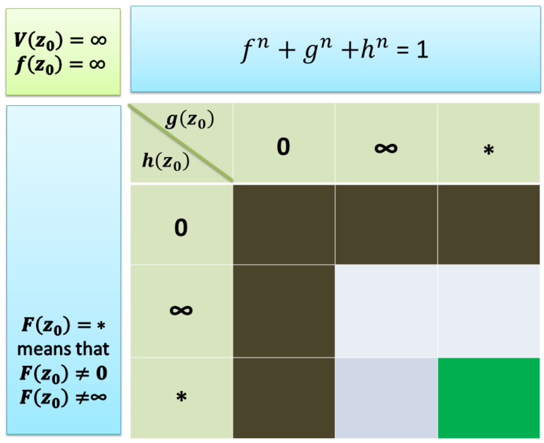
By the arguments above, we do not have to treat the case when z_0 is a zero of at least one of f, g and h.

Fig. 3.10: Estimates for poles and zeros



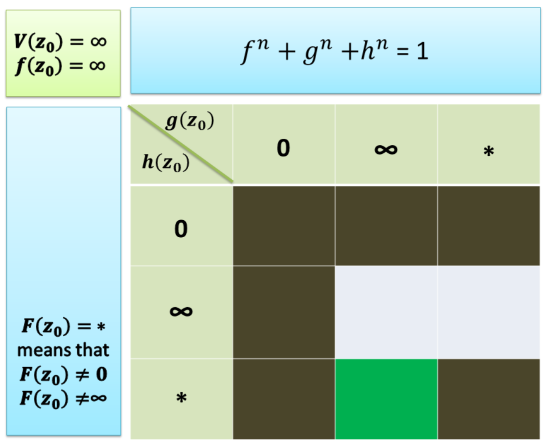
It is impossible that $z_0 \in (\mathbb{C} \setminus (\mathcal{Z}_g \bigcup \mathcal{P}_g)) \bigcap (\mathbb{C} \setminus (\mathcal{Z}_h \bigcup \mathcal{P}_h))$ by (3.1).

Fig. 3.11: Estimates for poles and zeros



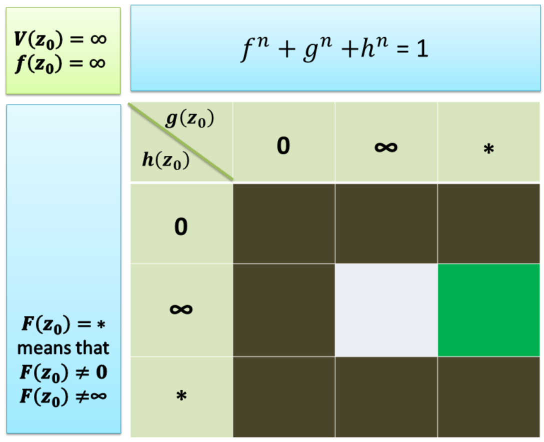
Suppose that $z_0 \in (\mathbb{C} \setminus (\mathcal{Z}_h \bigcup \mathcal{P}_h)) \bigcap \mathcal{P}_g$. We have $\omega(z_0, \eta_{gh}) \leq 2$ by (3.19). By (3.1), $\omega(z_0, f) = \omega(z_0, g)$, and hence g^2/f^{n-2} has a zero at z_0 of multiplicity $(n-4)\omega(z_0, f)$. By (3.19), we conclude that V has a zero at z_0 of multiplicity at least $(n-4)\omega(z_0, f) - 2$ since $n \geq 8$, a contradiction.

Fig. 3.12: Estimates for poles and zeros



This implies that $z_0 \in \mathcal{P}_h$. Similarly, changing roles of g and h, we conclude that V has a zero at z_0 of multiplicity at least that $(n-4)\omega(z_0, f) - 2$ when $z_0 \in (\mathbb{C} \setminus (\mathcal{Z}_g \bigcup \mathcal{P}_g)) \bigcap \mathcal{P}_h$ by (3.19). Hence, we showed the second assertion.

Fig. 3.13: Estimates for poles and zeros



Finally, we consider the case $z_0 \in \mathcal{P}_f \bigcap \mathcal{P}_g \bigcap \mathcal{P}_h$. We may set $d = \omega(z_0, f) = \omega(z_0, g) \geq \omega(z_0, h) = k$ by (3.1) without loss of generality.

In the case $d > k \geq 1$, by (3.19) we have $\omega(z_0, \eta_{gh}) \leq 3$. Since $n \geq 8$, by (3.19), we obtain that V has a zero at z_0 of multiplicity at least

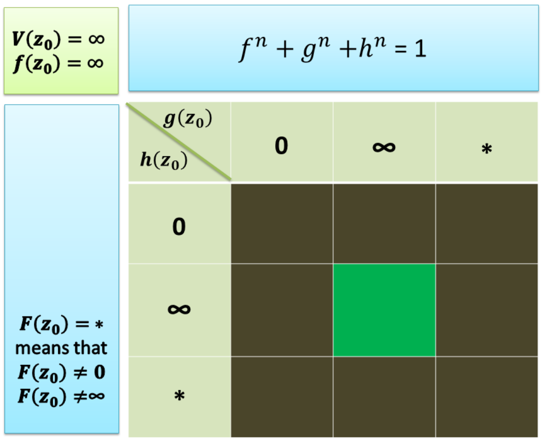
$$(n-2)d - (2d + 2k + 3) \geq (n-6)d - 1 \geq 2n - 13.$$

For the case $d = k \geq 1$, we note that $\omega(z_0, \eta_{gh}) \leq 2$ by the properties of Wronskian in the case $d = k$. By (3.19), $\omega(z_0, 1/V)$ can be estimates as

$$\omega\left(z_0, \frac{1}{V}\right) \geq (n-2)d - (2d + 2d + 2) = (n-6)d - 2. \tag{3.25}$$

By (3.25), if $d \geq 2$, then V has a zero at z_0 of multiplicity at least 2, since $n \geq 8$. If $d = 1$, V does not have neither a zero nor a pole at z_0 by (3.25). In particular, if $n \geq 9$, then V has a zero at z_0. We conclude that V cannot have a pole at z_0 when $z_0 \in \mathcal{P}_f \bigcap \mathcal{P}_g \bigcap \mathcal{P}_h$.

Fig. 3.14: Estimates for poles and zeros



Thus we have proved Lemma 3.3. $\square$

Remark 3.1 By the arguments in the proof of Lemma 3.3, we note that if $z_0 \in \mathcal{P}$, then z_0 is a zero of V for $n \geq 9$. Further, we have that z_0 is a zero of V for $n = 8$ if $z_0 \in \mathcal{P}$ of multiplicity at least 2.

Proof of Lemma 3.4 We may assume that $z_0 \in \mathcal{Z}_f$ and $\omega(z_0, 1/f) = d \geq 2$ without loss of generality. By (3.1), it is impossible that $z_0 \in \mathcal{Z}_g \bigcap \mathcal{Z}_h$. Hence, we may assume that $z_0 \notin \mathcal{Z}_g$ without loss of generality. By (3.20), we see that η_{hf} has a pole of multiplicity at most 3, and hence we have

$$\omega\left(z_0, \frac{1}{V}\right) \geq 2d - 3 \geq 1.$$

This shows that V has a zero at z_0. $\square$

3.5 Alternative Proofs of Known Results for Meromorphic Solutions

As we mentioned in Section 3.1, it is known that there do not exist solutions of non-constant rational functions to (3.1) when $n \geq 8$, and there do not exist transcendental meromorphic solutions to (3.1) when $n \geq 9$. Further, if there exist transcendental meromorphic solutions to (3.1) when $n = 8$, then they satisfy a differential equation. In this section, we give alternative proofs of these results and a slight improvement.

3.5.1 Solutions of Rational Functions

Here we consider solutions of rational functions to (3.1). We first show the following theorem.

Theorem 3.2 *Suppose that $n \geq 7$ in (3.1), and suppose that there exist non-constant rational functions f, g and h satisfying (3.1). Let V be a rational function given by (3.16). Then*

$$(n-6)(\mathfrak{m}(f) + \mathfrak{m}(g) + \mathfrak{m}(h)) \leq 3\left(\mathfrak{n}(V) - \mathfrak{n}\left(\frac{1}{V}\right)\right). \tag{3.26}$$

As a corollary to Theorem 3.2, we obtain the known result below.

Corollary 3.1 *There do not exist solutions of non-constant rational functions to (3.1) for $n \geq 8$.*

Proof of Theorem 3.2 Let R be an arbitrary rational function. We write $R = R_N/R_D$ with relatively prime polynomials R_N and R_D. Since $R'/R = R'_N/R_N - R'_D/R_D$, we see $\mathfrak{m}(R'/R) \leq \max\left(\mathfrak{m}(R'_N/R_N),\ \mathfrak{m}(R'_D/R_D)\right) = 0$ by Lemma 2.2 (ii), (i). Further, using this with Lemma 2.2 (ii), we obtain $\mathfrak{m}(R''/R) \leq \mathfrak{m}(R''/R') + \mathfrak{m}(R'/R) = 0$. Hence, by (3.8), and by (3.18) to (3.20), we have

$$\mathfrak{m}(\Delta) = 0, \quad \mathfrak{m}(\eta_{fg}) = \mathfrak{m}(\eta_{gh}) = \mathfrak{m}(\eta_{hf}) = 0. \tag{3.27}$$

Combining (3.21) and (3.22), we obtain

$$n^{2(n-6)}V^n = \Delta^{n-6}(\eta_{fg}\eta_{gh}\eta_{hf})^2. \tag{3.28}$$

By (3.28), Lemma 2.2 (ii) and (3.27), we have for $n \geq 7$

$$n\mathfrak{m}(V) \leq 2(\mathfrak{m}(\eta_{fg}) + \mathfrak{m}(\eta_{gh}) + \mathfrak{m}(\eta_{hf})) + (n-6)\mathfrak{m}(\Delta) = 0,$$

which gives

$$\mathfrak{m}(V) = 0. \tag{3.29}$$

By (3.19), Lemma 2.2 (ii) and (3.27), we have

$$\begin{aligned}(n-2)\mathfrak{m}(f) &\le 2\mathfrak{m}(g) + 2\mathfrak{m}(h) + \mathfrak{m}(\eta_{gh}) + \mathfrak{m}\left(\frac{1}{V}\right) \\ &= 2\mathfrak{m}(g) + 2\mathfrak{m}(h) + \mathfrak{m}\left(\frac{1}{V}\right). \end{aligned} \tag{3.30}$$

Similarly, by (3.18) and (3.20), we have

$$(n-2)\mathfrak{m}(h) \le 2\mathfrak{m}(f) + 2\mathfrak{m}(g) + \mathfrak{m}\left(\frac{1}{V}\right), \tag{3.31}$$

$$(n-2)\mathfrak{m}(g) \le 2\mathfrak{m}(h) + 2m(f) + \mathfrak{m}\left(\frac{1}{V}\right). \tag{3.32}$$

Combining (3.30) to (3.32), and using (2.3) and (3.29), we obtain

$$\begin{aligned}(n-6)(\mathfrak{m}(f) + \mathfrak{m}(g) + \mathfrak{m}(h)) &\le 3\mathfrak{m}\left(\frac{1}{V}\right) \\ &= 3\left(\mathfrak{n}(V) + \mathfrak{m}(V) - \mathfrak{n}\left(\frac{1}{V}\right)\right) = 3\left(\mathfrak{n}(V) - \mathfrak{n}\left(\frac{1}{V}\right)\right). \end{aligned}$$

We have thus proved Theorem 3.2. $\square$

Proof of Corollary 3.1 Assume that there exist rational functions f, g and h satisfying (3.1). By means of Lemma 3.3, we obtain $\mathfrak{n}(V) = 0$ since $n \ge 8$. By Theorem 3.2, we have

$$\mathfrak{m}(f) + \mathfrak{m}(g) + \mathfrak{m}(h) + \mathfrak{n}\left(\frac{1}{V}\right) = 0. \tag{3.33}$$

This implies that V is a constant, and that f, g and h have the property that the degree of the numerator is equal to or less than the degree of the denominator. This means that if $\mathcal{P} = \emptyset$ all of f, g and h reduce to constants, a contradiction.

We now consider the case $\mathcal{P} \ne \emptyset$. If f has a pole of multiplicity at least 2, then by Remark 3.1, V has a zero, a contradiction. Hence, f has no pole or each pole of f is a simple pole. Similarly, we see that g and h possess the same property. Since we assume $\mathcal{P} \ne \emptyset$, at least one of f, g and h admits a simple pole. On the other hand, for any entire function φ including a polynomial, $f(\varphi)$, $g(\varphi)$ and $h(\varphi)$ satisfy (3.1). Choosing a suitable polynomial φ, at least one of $f(\varphi)$, $g(\varphi)$ and $h(\varphi)$ admits a multiple pole, which is a contradiction. $\square$

3.5.2 Transcendental Meromorphic Solutions

From this section, we will utilize the Nevanlinna theory, which was mentioned in Chapter 2 Preliminaries items related to this dissertation.

Under the assumption that there exist transcendental meromorphic solutions f, g and h to (3.1), we write $T^*(r) = T(r, f) + T(r, g) + T(r, h)$, and denote by $S^*(r)$ any quantity that satisfies $S^*(r) = o(1)T^*(r)$ as $r \to \infty$, possibly outside a set of r of finite linear measure. Clearly if $\varphi(r) = S(r, f)$, $S(r, g)$ or $S(r, h)$, then $\varphi(r) = S^*(r)$. We call a meromorphic function a satisfying $T(r, a) = S^*(r)$ *small* with respect to f, g and h in this dissertation. We obtain the transcendental counterparts to Subsection 3.5.1.

Theorem 3.3 *Suppose that $n \geq 7$ in (3.1), and suppose that there exist non-constant meromorphic functions f, g and h satisfying (3.1), at least of which is transcendental. Let V be a meromorphic function given by (3.16). Then*

$$(n-6)(m(r,f) + m(r,g) + m(r,h)) \leq 3\left(N(r,V) - N\left(r, \frac{1}{V}\right)\right) + S^*(r). \tag{3.34}$$

As a corollary to Theorem 3.3, we obtain the result below.

Corollary 3.2 *There do not exist transcendental meromorphic solutions to (3.1) for $n \geq 9$. If there exist non-constant meromorphic functions f, g and h satisfying (3.1) for $n = 8$, at least of which is transcendental, then there exists a small entire function a with respect to f, g and h such that*

$$W(f^8, g^8, h^8) = af^6g^6h^6. \tag{3.35}$$

Proof of Theorem 3.3 We adopt the similar idea to the proof of Theorem 3.2. By the lemma on the logarithmic derivatives, we have

$$m(r, \Delta) = S^*(r), \ m(r, \eta_{fg}) = S^*(r), \ m(r, \eta_{gh}) = S^*(r), \ m(r, \eta_{hf}) = S^*(r). \tag{3.36}$$

$$m(r, V) = S^*(r). \tag{3.37}$$

In fact, by (3.28) and the lemma on the logarithmic derivatives, we have

$$nm(r, V) \leq 2(m(r, \eta_{fg}) + m(r, \eta_{gh}) + m(r, \eta_{hf})) + (n-6)m(r, \Delta) = S^*(r),$$

which shows that the assertion (3.37) holds. It follows from (3.19) and (3.36),

$$\begin{aligned}(n-2)m(r,f) &\leq 2m(r,g) + 2m(r,h) + m(r,\eta_{gh}) + m\left(r, \frac{1}{V}\right) + S^*(r) \\ &= 2m(r,g) + 2m(r,h) + m\left(r, \frac{1}{V}\right) + S^*(r).\end{aligned} \tag{3.38}$$

Similarly, by (3.18) and (3.20), we have

$$(n-2)m(r,h) \leq 2m(r,f) + 2m(r,g) + m\left(r, \frac{1}{V}\right) + S^*(r), \tag{3.39}$$

$$(n-2)m(r,g) \leq 2m(r,h) + 2m(r,f) + m\left(r, \frac{1}{V}\right) + S^*(r). \tag{3.40}$$

By (3.38) to (3.40), and by (3.37) and the first main theorem due to Nevanlinna, we obtain

$$\begin{aligned}
&(n-6)(m(r,f) + m(r,g) + m(r,h)) \\
&\quad \leq 3m\left(r, \frac{1}{V}\right) + S^*(r) = 3\left(N(r,V) - N\left(r, \frac{1}{V}\right)\right) + S^*(r).
\end{aligned}$$

Hence we proved Theorem 3.3. □

Proof of Corollary 3.2 Assume that there exist transcendental meromorphic functions f, g and h satisfying (3.1). By means of Lemma 3.3, we see that V is an entire function, and hence $N(r,V) = S^*(r)$. Combining this and (3.37), we obtain that $T(r,V) = S^*(r)$, which shows that V is a small entire function with respect to f, g and h. In particular, this implies that f, g and h satisfy the differential equation (3.16) with a small entire function V for $n = 8$. Setting $a = 64V$ in (3.16), we obtain (3.35).

We consider below the case $n \geq 9$. By means of Theorem 3.3 and Lemma 3.3, we have

$$m(r,f) + m(r,g) + m(r,h) = S^*(r). \tag{3.41}$$

By Remark 3.1, if $z_0 \in \mathcal{P}$ then z_0 is a zero of V. We set $\Phi(d) = d/((n-6)d-2)$, $d \geq 1$, which is a decreasing function in d. Using the arguments in the proof of Lemma 3.3, we have

$$\omega(z_0, f) \leq \Phi(\omega(z_0, f))\omega\left(z_0, \frac{1}{V}\right) \leq \frac{1}{n-8}\omega\left(z_0, \frac{1}{V}\right), \tag{3.42}$$

which yields $N(r,f) \leq N(r,1/V) \leq T(r,V) + S^*(r)$. Since V is a small function with respect to f, g and h, we obtain $N(r,f) = S^*(r)$. Similarly, we obtain $N(r,g) = S^*(r)$ and $N(r,h) = S^*(r)$. Combining these estimates with (3.41), we conclude that $T(r,f) = S^*(r)$, $T(r,g) = S^*(r)$ and $T(r,h) = S^*(r)$, a contradiction. Thus, for $n \geq 9$ there do not exist transcendental meromorphic solutions to (3.1). □

In [16], Ishizaki showed that (3.35) holds with a small meromorphic function a. Corollary 3.2 says that a does not have any poles.

3.6 Alternative Proofs of Known Results for Entire Solutions

3.6.1 Polynomial Solutions

We adopt similar methods to those used in Subsection 3.5.1 to obtain the known result below.

Corollary 3.3 *There do not exist nonconstant polynomial solutions to* (3.1) *for* $n \geq 6$.

Proof of Corollary 3.3 Assume that there exist polynomials f, g and h satisfying (3.1). When $n \geq 7$, we apply Theorem 3.2 to polynomial solutions noting that $\mathfrak{m}(f) = \deg f$, $\mathfrak{m}(g) = \deg g$ and $\mathfrak{m}(h) = \deg h$ to obtain

$$(n-6)(\deg f + \deg g + \deg h) \leq 3\mathfrak{n}(V). \tag{3.43}$$

Since f, g and h are polynomials, V is a polynomial given by (3.16). This implies that $\mathfrak{n}(V) = 0$. By (3.43), $\deg f + \deg g + \deg h = 0$. This means that all of f, g and h are constants, a contradiction.

We consider the case $n = 6$. It follows from (3.22),

$$V^3 = \eta_{fg}\eta_{gh}\eta_{hf}. \tag{3.44}$$

We note that (3.27) holds for $n \geq 2$. By (3.44) and (3.27), we obtain

$$\deg V = \mathfrak{m}(V) = 0, \tag{3.45}$$

which implies that V reduces to a constant.

If $\mathcal{Z} = \emptyset$, all of f, g and h reduce to constants, a contradiction. We consider the case $\mathcal{Z} \neq \emptyset$. If f has a zero of multiplicity at least 2, then Lemma 3.4, V has a zero, a contradiction. Hence, f has at least one simple zero. Similarly, we see that g and h possess the same property. On the other hand, for any entire function φ including a polynomial, $f(\varphi)$, $g(\varphi)$ and $h(\varphi)$ satisfy (3.1). Choosing a suitable polynomial φ, at least one of $f(\varphi)$, $g(\varphi)$ and $h(\varphi)$ admits a multiple zero, which is a contradiction. $\square$

3.6.2 Transcendental Entire Solutions

As a corollary to Theorem 3.3, we obtain the known result for transcendental entire solutions below.

Corollary 3.4 *There do not exist transcendental entire solutions to* (3.1) *for* $n \geq 7$. *If there exist transcendental entire functions* f, g *and* h *satisfying* (3.1) *for* $n = 6$, *then there exists a small entire function* b *with respect to* f, g *and* h *such that*

$$W(f^6, g^6, h^6) = bf^4g^4h^4. \tag{3.46}$$

Proof of Corollary 3.4 Assume that there exist transcendental entire functions f, g and h satisfying (3.1) for $n \geq 7$. Since V is entire, by Theorem 3.3, we have $m(r, f) + m(r, g) + m(r, h) = S^*(r)$. Since f, g and h are entire, we have $T(r, f) = S^*(r)$, $T(r, g) = S^*(r)$ and $T(r, h) = S^*(r)$, a contradiction. Hence there do not exist transcendental entire solutions to (3.1) for $n \geq 7$.

Concerning the case $n = 6$, by (3.44) and (3.36), we have

$$3m(r, V) \leq m(r, \eta_{fg}) + m(r, \eta_{gh}) + m(r, \eta_{hf}) = S^*(r),$$

which shows that $m(r, V) = S^*(r)$. Since V is entire, we have $T(r, V) = S^*(r)$. Hence V is a small entire function with respect to f, g and h. Setting $b = 36V$ in (3.16), we see that the assertion follows. □

3.7 Degrees of Polynomial Solutions

We discuss degrees of polynomial solutions to (3.1) below. Suppose that there exist polynomials f, g and h satisfy (3.1). Clearly, $f(P)$, $g(P)$ and $h(P)$ are polynomial solutions to (3.1) for any polynomial P. It implies that there is no upper bound of degrees of polynomial solutions. For the lower bound, we have the following result.

Proposition 3.1 *Let* $2 \leq n \leq 5$ *be an integer. Suppose that there exist non-constant polynomials* f, g *and* h *satisfying* (3.1)*, and assume* $d = \deg f = \deg g \geq \deg h = k$*. Then*

(i) *For the case* $d = k$*, we have* $d \geq 4/(6-n)$.

(ii) *For the case* $d > k$*, we have*
$d \geq 2$ *and* $k \geq 1$ *when* $n = 2, 3$,
$d \geq 3$ *and* $k \geq 2$ *when* $n = 4$,
$d \geq 5$ *and* $k \geq 4$ *when* $n = 5$.

Regarding (ii), we improve it in Proposition 4.1 mentioned later.

Example 3.4 We recall examples for $n = 2$ and $n = 3$

$$\left(\frac{1+P(z)}{\sqrt{2}}\right)^2 + \left(\frac{1-P(z)}{\sqrt{2}}\right)^2 + (iP(z))^2 = 1, \tag{3.47}$$

and

$$\left(\frac{1+P(z)^3}{\sqrt[3]{2}}\right)^3 + \left(\frac{1-P(z)^3}{\sqrt[3]{2}}\right)^3 + (\omega\sqrt[3]{3}P(z)^2)^3 = 1, \tag{3.48}$$

where $P(z)$ is an arbitrary polynomial and $\omega^3 = -1$, see e.g. [9], [13], [22].

This example shows that both cases $\deg f = \deg g = \deg h$ and $\deg f = \deg g > \deg h$ occur in general.

We see that Proposition 3.1 follows immediately from the following lemma.

Lemma 3.5 *Suppose that there exist non-constant polynomials* f, g *and* h *satisfying* (3.1)*, and assume* $d = \deg f = \deg g \geq \deg h = k$*. Let* v *be the degree of* V *defined in* (3.16).

(i) *For the case* $d = k$*, we have*

$$(6-n)d \geq 4 + v. \tag{3.49}$$

(ii) *For the case* $d > k$, *we have*

$$(4-n)d + 2k = 3 + v. \tag{3.50}$$

The method for the proof of Lemma 3.5 is similar to the idea in [24]. Since the situations are different, we give the proof.

Proof of Lemma 3.5

(i) First we assume that $\deg f = \deg g = \deg h$.

Write

$$\begin{aligned} f(z) &= a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0, && a_d \neq 0, \\ g(z) &= b_d z^d + b_{d-1} z^{d-1} + \cdots + b_0, && b_d \neq 0. \end{aligned}$$

Then

$$\begin{aligned} f'(z) &= d a_d z^{d-1} + (d-1) a_{d-1} z^{d-2} + \cdots + a_1, \\ g'(z) &= d b_d z^{d-1} + (d-1) b_{d-1} z^{d-2} + \cdots + b_1, \\ f''(z) &= d(d-1) a_d z^{d-2} + (d-1)(d-2) a_{d-1} z^{d-3} + \cdots + a_2, \\ g''(z) &= d(d-1) b_d z^{d-2} + (d-1)(d-2) b_{d-1} z^{d-3} + \cdots + b_2. \end{aligned}$$

So, we can calculate as follows.

$$\begin{aligned} W(f,g) &= \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g \\ &= d a_d b_d z^{d+d-1} + (d-1) a_d b_{d-1} z^{d+d-2} + \cdots \\ &\quad - (d a_d b_d z^{d+d-1} + (d-1) a_{d-1} b_d z^{d+d-2} + \cdots) \\ &= (d-1)(a_d b_{d-1} - a_{d-1} b_d) z^{2d-2} + \cdots \\ W(f',g') &= \begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix} = f'g'' - f''g' \\ &= d^2(d-1) a_d b_d z^{(d-1)+(d-2)} + d(d-1)^2 a_{d-1} b_d z^{(d-2)+(d-2)} + \cdots \\ &\quad - (d^2(d-1) a_d b_d z^{(d-1)+(d-2)} + d(d-1)^2 a_d b_{d-1} z^{(d-2)+(d-2)} + \cdots) \\ &= d(d-1)^2 (a_{d-1} b_d - a_d b_{d-1}) z^{2d-4} + \cdots . \end{aligned}$$

Then we see that $\deg W(f,g)$ is at most $2d-2$, and $\deg W(f',g')$ is at most $2d-4$.

By (3.12) and (3.13),we have follows.

$$\begin{aligned} W(F,G,H) &= W(f^n, g^n, h^n) \\ &= n^2 f^{n-2} g^{n-2} V_1 \\ &= n^2 f^{n-2} g^{n-2} \Big((n-1) f'g' W(f,g) + fg W(f'g') \Big). \end{aligned}$$

Hence, $\deg W(F,G,H)$ is at most $2nd - 4$.
By (3.16), we have

$$2nd - 4 \geq \deg W(F,G,H) = \deg(f^{n-2}g^{n-2}h^{n-2}V) = 3d(n-2) + v,$$

and hence (3.49) follows. We have thus proved (i).

(ii) We next consider the case $k = \deg h < d$. Then $\deg W(h,f) = d + k - 1$, and $\deg W(h',f') = d + k - 3$. By (3.15), we have that $\deg W(F,G,H)$ is at most $n(d+k) - 3$. Further, we investigate $\deg W(F,G,H)$ in terms of (3.15) in detail. Write

$$f(z) = a_d z^d + a_{d-1}z^{d-1} + \cdots + a_0, \quad a_d \neq 0, \tag{3.51}$$

$$h(z) = c_k z^k + c_{k-1}z^{k-1} + \cdots + c_0, \quad c_k \neq 0. \tag{3.52}$$

Using (3.51) and (3.52), we compute

$$(n-1)h'f'W(h,f) + hfW(h',f') = n(d-k)dka_d^2c_k^2\, z^{2d+2k-3} + \cdots + C,$$

where C is a constant. This gives that $\deg W(F,G,H) = n(d+k) - 3$.

By means of (3.16), we obtain $n(d+k) - 3 = 2d(n-2) + k(n-2) + v$, that is, (3.50). Hence, (ii) is proved. $\square$

Remark 3.2 If $n \geq 6$, it is impossible to exist polynomial solutions to (3.1) for both cases $d = k$ and $d > k$ by (3.49) and (3.50). This yields an alternative proof of Corollary 3.3.

Example 3.5 We compute $\deg W(F,G,H)$ and $\deg V$ of the examples given in (3.47) and (3.48).

For (3.47), we obtain $W(F,G,H) = 2P'^3$ and $V(z) = \frac{1}{2}P'^3$ by simple computations. Setting $P(z) = z$ for simplicity, we have $W(F,G,H) = 4V = 2$. In this case, the equality in (3.49) holds with $n = 2$, $d = k = 1$ and $v = 0$. This example shows that the estimate (3.49) in Lemma 3.5 is sharp.

We consider (3.48) nothing that $d > k$. We have $W(F,G,H) = -243P^6(-1+P^6)P'^3$, and $V = (9\sqrt[3]{6^2}P^4P'^3)/\omega$. Setting $P(z) = z$ for simplicity, we have $W(F,G,H) = -243z^6(-1+z^6)$, and $V(z) = (9\sqrt[3]{6^2}z^4)/\omega$. We see that equality in (3.50) holds, since $n = 3$, $d = 3$, $k = 2$ and $v = 4$ in this case.

Chapter 4

Case General

4.1 Background of Case General

In this chapter, we are concerned with the general problem of determining whether or not there exist non-constant meromorphic functions $f_1, f_2, \ldots, f_k$ which satisfy the Fermat type functional equation

$$f_1^n + f_2^n + \cdots + f_k^n = 1, \tag{4.1}$$

for a given pair of positive integers $\{n, k\}$. We consider this problem for the following four classes of functions: meromorphic functions, rational functions, entire functions, polynomials. For each of these classes of functions, there exist pairs of positive integers $\{n, k\}$ (a) where (4.1) does not possess non-constant solutions, (b) where (4.1) possesses non-constant solutions, and (c) where it is not known whether (4.1) possesses non-constant solutions; see [9], [13]. It is organized as follows.

	If solutions exist
Meromorphic functions	$n \leq k^2 - 1$
Rational functions	$n \leq k^2 - 2$
Entire functions	$n \leq k^2 - k$
Polynomials	$n \leq k^2 - k - 1$

In the case of $k = 2$, it is clear where the solution exists in the four functions.

	If solutions exist
Meromorphic functions	$n \leq k^2 - 1 = 3$
Rational functions	$n \leq k^2 - 2 = 2$
Entire functions	$n \leq k^2 - k = 2$
Polynomials	$n \leq k^2 - k - 1 = 1$

Example 4.1 *We recall examples for $n = 2$ and $n = 3$ of $f_1^n + f_2^n = 1$.*

$$\sin^2 w(z) + \cos^2 w(z) = 1,$$

where $w(z)$ is an arbitrary entire function.

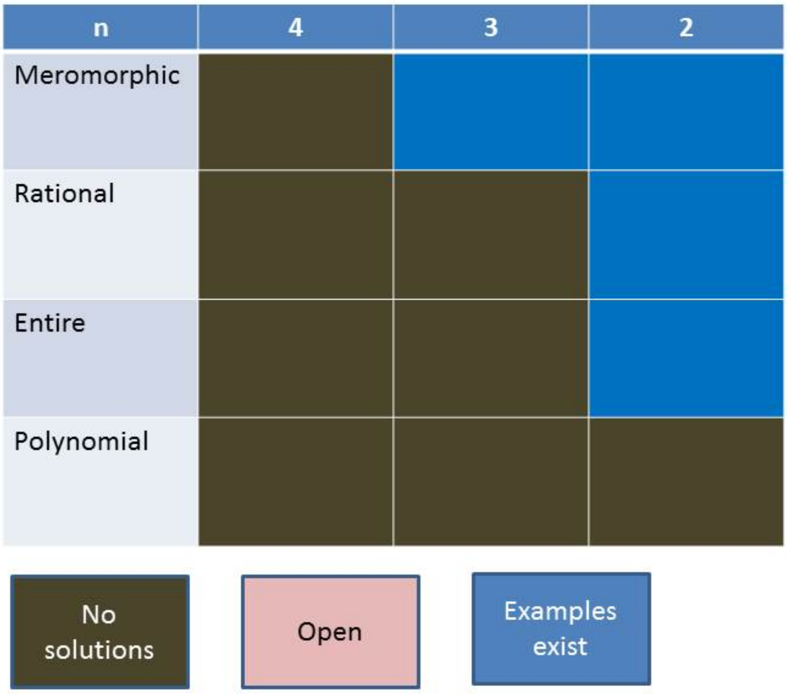
$$\left(\frac{1 + w(z)^2}{2w(z)}\right)^2 + \left(i\,\frac{1 - w(z)^2}{2w(z)}\right)^2 = 1,$$

where $w(z)$ is an arbitrary meromorphic function.

$$\left(\frac{1 + c\wp'(z)}{2\wp(z)}\right)^3 + \left(\frac{1 - c\wp'(z)}{2\wp(z)}\right)^3 = 1,$$

where $\wp(z)$ is an elliptic function satisfying $(\wp')^2 = 4\wp^3 - 1$, and $c = \dfrac{1}{\sqrt{3}}$.

Fig. 4.1: Known Results for $f^n + g^n = 1$



In Chapter 3 we considered the case $k = 3$ and showed the status of the solution in Figure 3.1. The open questions for those $\{n, k\}$ in (c) have attracted increasing interest. The main purpose of this chapter is to give, for each class of functions, necessary conditions that non-constant solutions of (4.1) where $\{n, k\}$ is in (c), would have to satisfy.

Section 4.3 contains the main results for rational functions and transcendental meromorphic functions, while corollaries and observations on transcendental entire functions and polynomials are in Sections 4.4 and 4.5. Section 4.2 contain observations about rational functions and calculations with Wronskians of meromorphic functions, which we use to prove our results. Last, we discuss the case when $k = 4$ in (4.1) in Section 4.6.

In this chapter as well as the previous chapter, we use standard notations in the Nevanlinna theory described in Preliminaries. Under the assumption that there exist transcendental meromorphic solutions f_1, f_2, ..., f_k to (4.1), we write $T^*(r) = \sum_{j=1}^{k} T(r, f_j)$, and denote by $S^*(r)$ any quantity that satisfies $S^*(r) = o(1)T^*(r)$ as $r \to \infty$, possibly outside a set of r of finite linear measure. Clearly, if $\psi(r) = S(r, f_j)$ for some $j \in \{1, 2, \ldots, k\}$, then $\psi(r) = S^*(r)$. We call a meromorphic function a satisfying $T(r, a) = S^*(r)$ *small* with respect to f_1, f_2, ..., f_k.

4.2 Calculations with Wronskians

We assume that there exist non-constant meromorphic functions f_1, f_2, $\ldots$, f_k which satisfy the functional equation (4.1) where n and k are fixed integers satisfying $n \geq k-1$ and $k \geq 2$. For the sake of simplicity we put $f_1^n = F_1$, $f_2^n = F_2$, $\ldots$, $f_k^n = F_k$, and define

$$\begin{aligned} W &= W(f_1^n, f_2^n, \ldots, f_k^n) \\ &= W(F_1, F_2, \ldots, F_k) \\ &= \begin{vmatrix} F_1 & F_2 & \ldots & F_k \\ F_1' & F_2' & \ldots & F_k' \\ \vdots & \vdots & \ddots & \vdots \\ F_1^{(k-1)} & F_2^{(k-1)} & \ldots & F_k^{(k-1)} \end{vmatrix}. \end{aligned} \tag{4.2}$$

First using (4.1), we eliminate f_k and F_k from (4.2),

$$\begin{aligned} W &= \begin{vmatrix} F_1 & F_2 & \ldots & F_{k-1} & 1 \\ F_1' & F_2' & \ldots & F_{k-1}' & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_1^{(k-1)} & F_2^{(k-1)} & \ldots & F_{k-1}^{(k-1)} & 0 \end{vmatrix} \\ &= (-1)^{k-1} \begin{vmatrix} F_1' & F_2' & \ldots & F_{k-1}' \\ \vdots & \vdots & \ddots & \vdots \\ F_1^{(k-1)} & F_2^{(k-1)} & \ldots & F_{k-1}^{(k-1)} \end{vmatrix}. \end{aligned} \tag{4.3}$$

By the Leibniz formula, for any meromorphic function φ, we have $(\varphi^n)' = \varphi^{n-k+1}U_1(\varphi)$, $(\varphi^n)'' = \varphi^{n-k+1}U_2(\varphi)$, $\ldots$, $(\varphi^n)^{(k-1)} = \varphi^{n-k+1}U_{k-1}(\varphi)$, where $U_j(\varphi)$, $j = 1, 2, \ldots, k-1$ are homogeneous differential polynomials in φ of degree $k-1$.
Calculation examples as follows.

When $n = 6$ and $k = 3$, $\quad (\varphi^6)'' = \varphi^4 \cdot (6(5(\varphi')^2 + \varphi\varphi''))$.

When $n = 12$ and $k = 4$, $\quad (\varphi^{12})''' = \varphi^9 \cdot (12(110(\varphi')^3 + 33\varphi\varphi'\varphi'' + \varphi^2\varphi'''))$.

This implies that if $\varphi \not\equiv 0$ is a rational function, then from (2.4) and (2.5),

$$\mathfrak{m}\left(\frac{U_j(\varphi)}{\varphi^{k-1}}\right) = 0, \quad j = 1, 2, \ldots, k-1, \tag{4.4}$$

and if φ is a transcendental meromorphic function, by the theorem on the logarithmic derivatives,

$$m\left(r, \frac{U_j(\varphi)}{\varphi^{k-1}}\right) = S(r, \varphi), \quad j = 1, 2, \ldots, k-1. \tag{4.5}$$

Using (4.3) we write W as follows

$$W = (-1)^{k-1}(f_1 f_2 \cdots f_{k-1})^{n-k+1}\begin{vmatrix} U_1(f_1) & U_1(f_2) & \dots & U_1(f_{k-1}) \\ \vdots & \vdots & \ddots & \vdots \\ U_{k-1}(f_1) & U_{k-1}(f_2) & \dots & U_{k-1}(f_{k-1}) \end{vmatrix}$$
$$= V_k \prod_{\substack{1\le j\le k \\ j\ne k}} f_j^{n-k+1}. \tag{4.6}$$

Similar to (4.6), using (4.1), we eliminate f_m and F_m, $1 \le m \le k$ from (4.2),

$$W = V_m \prod_{\substack{1\le j\le k \\ j\ne m}} f_j^{n-k+1}, \tag{4.7}$$

where

$$V_m = (-1)^{m-1}\begin{vmatrix} U_1(f_1) & \dots & U_1(f_{m-1}) & U_1(f_{m+1}) & \dots & U_1(f_k) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ U_{k-1}(f_1) & \dots & U_{k-1}(f_{m-1}) & U_{k-1}(f_{m+1}) & \dots & U_{k-1}(f_k) \end{vmatrix}. \tag{4.8}$$

We assert that if $f_1, f_2, \dots, f_k$ are rational functions, then

$$\mathfrak{m}\left(\frac{V_m}{\prod_{\substack{1\le j\le k \\ j\ne m}} f_j^{k-1}}\right) = 0, \quad 1 \le m \le k, \tag{4.9}$$

and if $f_1, f_2, \dots, f_k$ are transcendental meromorphic functions, then

$$m\left(r, \frac{V_m}{\prod_{\substack{1\le j\le k \\ j\ne m}} f_j^{k-1}}\right) = S^*(r), \quad 1 \le m \le k. \tag{4.10}$$

In fact, it follows from (4.8) that

$$\frac{V_m}{\prod_{\substack{1\le j\le k \\ j\ne m}} f_j^{k-1}} = (-1)^{m-1}\begin{vmatrix} \frac{U_1(f_1)}{f_1^{k-1}} & \dots & \frac{U_1(f_{m-1})}{f_{m-1}^{k-1}} & \frac{U_1(f_{m+1})}{f_{m+1}^{k-1}} & \dots & \frac{U_1(f_k)}{f_k^{k-1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{U_{k-1}(f_1)}{f_1^{k-1}} & \dots & \frac{U_{k-1}(f_{m-1})}{f_{m-1}^{k-1}} & \frac{U_{k-1}(f_{m+1})}{f_{m+1}^{k-1}} & \dots & \frac{U_{k-1}(f_k)}{f_k^{k-1}} \end{vmatrix}. \tag{4.11}$$

By (4.11), (4.4) and (4.5), the assertions (4.9) and (4.10) are ascertained.

We now let V denote the function

$$V = \frac{W}{\prod_{j=1}^{k} f_j^{n-k+1}}. \tag{4.12}$$

If $f_1, f_2, \ldots, f_k$ are all entire functions, then V in (4.12) is an entire function. In particular, if $f_1, f_2, \ldots, f_k$ are all polynomials, then V is a polynomial.

Lemma 4.1 *Suppose that $f_1, f_2, \ldots, f_k$ are non-constant meromorphic functions satisfying (4.1), where $n \geq k(k-1)$. Then we obtain the following estimates.*

(i) *If $f_1, f_2, \ldots, f_k$ are rational functions, then $\mathfrak{m}(V) = 0$.*

(ii) *If $f_1, f_2, \ldots, f_k$ are transcendental, then $m(r, V) = S^*(r)$.*

Proof of Lemma 4.1 Since the hypothesis implies that $n \geq k-1$ and $k \geq 2$, we can use the calculations above. From (4.7) and (4.12), we obtain

$$V_m = f_m^{n-k+1} V, \quad 1 \leq m \leq k, \tag{4.13}$$

and hence

$$V = \frac{V_m}{f_m^{n-k+1}} = \frac{\prod_{\substack{1\leq j\leq k\\ j\neq m}} f_j^{k-1}}{f_m^{n-k+1}} \eta_m, \quad \text{with} \quad \eta_m = \frac{V_m}{\prod_{\substack{1\leq j\leq k\\ j\neq m}} f_j^{k-1}}, \quad 1 \leq m \leq k. \tag{4.14}$$

Note that η_m, $1 \leq m \leq k$ are given by (4.11) and (4.14), and have properties (4.9) and (4.10). It follows from (4.14) that

$$V^k = \prod_{m=1}^{k} \left(\frac{\prod_{\substack{1\leq j\leq k\\ j\neq m}} f_j^{k-1}}{f_m^{n-k+1}} \eta_m \right) = \frac{\prod_{j=1}^k f_j^{(k-1)^2}}{\prod_{j=1}^k f_j^{n-k+1}} \prod_{j=1}^{k} \eta_j = \frac{\prod_{j=1}^k \eta_j}{\displaystyle\prod_{j=1}^{k} f_j^{n-(k-1)-(k-1)^2}},$$

namely,

$$V^k \prod_{j=1}^{k} f_j^{n-k(k-1)} = \prod_{j=1}^{k} \eta_j. \tag{4.15}$$

Define Δ by

$$\Delta = \frac{W}{\prod_{j=1}^k f_j^n} = \frac{W(F_1, F_2, \ldots, F_k)}{F_1 F_2 \cdots F_k} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{F_1'}{F_1} & \frac{F_2'}{F_2} & \cdots & \frac{F_k'}{F_k} \\ \frac{F_1''}{F_1} & \frac{F_2''}{F_2} & \cdots & \frac{F_k''}{F_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{F_1^{(k-1)}}{F_1} & \frac{F_2^{(k-1)}}{F_2} & \cdots & \frac{F_k^{(k-1)}}{F_k} \end{vmatrix}. \tag{4.16}$$

Combining (4.12) and (4.16), we have

$$\Delta = \frac{W}{\prod_{j=1}^{k} f_j^n} = \frac{V\prod_{j=1}^{k} f_j^{n-k+1}}{\prod_{j=1}^{k} f_j^n} = V\prod_{j=1}^{k} f_j^{-k+1},$$

namely,

$$\Delta\prod_{j=1}^{k} f_j^{k-1} = V. \tag{4.17}$$

It follows from (4.15) and (4.17) that

$$V^n = \Delta^{n-k(k-1)}\prod_{j=1}^{k} \eta_j^{k-1}. \tag{4.18}$$

By (4.18), (4.16) and (4.9), we have $\mathfrak{m}\,(V) = 0$ if $f_1, f_2, \ldots, f_k$ are rational functions, which implies Lemma 4.1 (i). By (4.18), (4.16) and (4.10), we have $m\,(r, V) = S^*(r)$ if $f_1, f_2, \ldots, f_k$ are transcendental, which implies Lemma 4.1 (ii). $\square$

Lemma 4.2 *Suppose that $f_1, f_2, \ldots, f_k$ are non-constant entire functions satisfying (4.1), where $n \geq k(k-1)$ and $k \geq 3$. Further, suppose that z_0 is a zero of at least one of $f_1, f_2, \ldots, f_k$ with multiplicity at least $k-1$. Then V has a zero at z_0, where V is the entire function given by (4.12).*

Let w be a meromorphic function. If z_0 is a pole of multiplicity μ (≥ 1) for $w(z)$, then we set $p(z_0, w) = \mu$, and if $w(z_0) \neq \infty$, then we set $p(z_0, w) = 0$.

Proof of Lemma 4.2 We may assume that z_0 is a zero of some f_ℓ, $1 \leq \ell \leq k$ where $p(z_0, 1/f_\ell) \geq k-1$. By (4.1), it is impossible to have $f_j(z_0) = 0$ for all $j = 1, 2, \ldots, k$. Hence, there exists an integer m satisfying $1 \leq m \leq k$ and $m \neq \ell$ such that $f_m(z_0) \neq 0$. Let η_m be the meromorphic function given by (4.14) and (4.11). Using the properties of U_j, $1 \leq j \leq k$, $j \neq m$ and (4.11), it can be deduced that η_m has a pole of multiplicity at most $\sum_{j=1}^{k-1} j = \frac{k(k-1)}{2}$. By (4.14), the multiplicity of the zero at z_0 of V can be estimated as

$$p\left(z_0, \frac{1}{V}\right) \geq p\left(z_0, \frac{1}{f_\ell^{k-1}}\right) - p\left(z_0, \eta_m\right) \geq (k-1)^2 - \frac{k(k-1)}{2} = \frac{1}{2}(k-1)(k-2) \geq 1.$$

This shows that V has a zero at z_0. $\square$

4.3 Main Results

We first consider solutions to (4.1) in the class of rational functions.

Theorem 4.1 *Suppose that there exist non-constant rational functions $f_1, f_2, \ldots, f_k$ satisfying (4.1), where $n \geq k(k-1)+1$. Let V be the rational function given by (4.12). Then*

$$(n-k(k-1))\sum_{j=1}^{k}\mathfrak{m}(f_j) \leq k\left(\mathfrak{n}(V)-\mathfrak{n}\left(\frac{1}{V}\right)\right). \tag{4.19}$$

Proof of Theorem 4.1 Since the hypothesis implies that $n \geq k-1$ and $k \geq 2$, the calculations in Section 4.2 can be used. We write (4.14) as

$$f_m^{n-k+1} = \frac{\prod_{\substack{1\leq j\leq k \\ j\neq m}} f_j^{k-1}}{V}\,\eta_m, \quad 1 \leq m \leq k. \tag{4.20}$$

By (4.20), (4.9) and (2.5), we have

$$\begin{aligned}(n-k+1)\mathfrak{m}(f_m) &\leq (k-1)\sum_{\substack{1\leq j\leq k \\ j\neq m}}\mathfrak{m}(f_j)+\mathfrak{m}(\eta_m)+\mathfrak{m}\left(\frac{1}{V}\right) \\ &= (k-1)\sum_{\substack{1\leq j\leq k \\ j\neq m}}\mathfrak{m}(f_j)+\mathfrak{m}\left(\frac{1}{V}\right), \quad 1 \leq m \leq k. \end{aligned} \tag{4.21}$$

It follows from (4.21) that

$$\begin{aligned}(n-k+1)\sum_{m=1}^{k}\mathfrak{m}(f_m) &\leq (k-1)\sum_{m=1}^{k}\sum_{\substack{1\leq j\leq k \\ j\neq m}}\mathfrak{m}(f_j)+k\mathfrak{m}\left(\frac{1}{V}\right) \\ &= (k-1)^2\sum_{j=1}^{k}\mathfrak{m}(f_j)+k\mathfrak{m}\left(\frac{1}{V}\right),\end{aligned}$$

which gives

$$(n-k(k-1))\sum_{j=1}^{k}\mathfrak{m}(f_j) \leq k\mathfrak{m}\left(\frac{1}{V}\right). \tag{4.22}$$

By means of (2.2), (2.3) and (4.22), we obtain

$$(n-k(k-1))\sum_{j=1}^{k}\mathfrak{m}(f_j) \leq k\left(\mathfrak{n}(V)+\mathfrak{m}(V)-\mathfrak{n}\left(\frac{1}{V}\right)\right). \tag{4.23}$$

By (4.23) and Lemma 4.1 (i), we obtain (4.19). We have thus proved Theorem 4.1. □

Secondly, we consider transcendental meromorphic solutions to (4.1).

Theorem 4.2 *Suppose that there exist transcendental meromorphic functions f_1, f_2, ..., f_k satisfying* (4.1) *where $n \geq k(k-1)+1$. Let V be the meromorphic function given by* (4.12). *Then*

$$(n-k(k-1))\sum_{j=1}^{k} m(r,f_j) \leq k\left(N(r,V) - N\left(r,\frac{1}{V}\right)\right) + S^*(r). \tag{4.24}$$

Proof of Theorem 4.2 Using the similar arguments in the proof of Theorem 4.1, we obtain the proof of Theorem 4.2 by replacing $\mathfrak{m}(\cdot)$ with $m(r,\cdot)$, replacing (2.3) with the first fundamental theorem of Nevanlinna, and replacing Lemma 4.1 (i) with Lemma 4.1 (ii). □

4.4 Transcendental Entire Solutions

As an application of Theorem 4.2, we obtain the following corollary, which is a generalization of [16, Proposition 6.1] and Corollary 3.4.

Corollary 4.1 *There do not exist transcendental entire solutions to* (4.1) *for* $n \geq k(k-1)+1$. *If there exist transcendental entire functions* $f_1, f_2, \ldots, f_k$ *satisfying* (4.1) *for* $n = k(k-1)$*, then there exists a small entire function* b *with respect to* $f_1, f_2, \ldots, f_k$ *such that*

$$W(f_1^{k(k-1)}, f_2^{k(k-1)}, \ldots, f_k^{k(k-1)}) = b \prod_{j=1}^{k} f_j^{(k-1)^2}. \tag{4.25}$$

Proof of Corollary 4.1 Assume that there exist transcendental entire solutions $f_1, f_2, \ldots, f_k$ to (4.1) for $n \geq k(k-1)+1$. Since V is entire in this case, we have $N(r,V) = 0$. By means of Theorem 4.2, we have $\sum_{j=1}^{k} m(r, f_j) = \sum_{j=1}^{k} T(r, f_j) = S^*(r)$, a contradiction. Hence, there do not exist transcendental entire solutions to (4.1) for $n \geq k(k-1)+1$.

Next we assume that there exist transcendental entire solutions $f_1, f_2, \ldots, f_k$ to (4.1) for $n = k(k-1)$. By (4.15), (4.14) and (4.10), we have

$$km(r,V) \leq \sum_{j=1}^{k} m(r, \eta_j) = S^*(r),$$

which shows that $m(r,V) = S^*(r)$. Since V is entire, we have $T(r,V) = S^*(r)$. Hence V is a small entire function with respect to $f_1, f_2, \ldots, f_k$. Setting $b = V$ in (4.12) and noting $k(k-1) - k + 1 = (k-1)^2$, we see that (4.25) holds. □

If γ is a nonconstant entire function, then $\sin^2 \gamma + \cos^2 \gamma = 1$, which gives an example of (4.25) with $n = k = 2$ and $b = -2\gamma'$. It is an open question whether there exist transcendental entire functions $f_1, f_2, \ldots, f_k$ that satisfy (4.1) when $n = k(k-1)$ and $k \geq 3$, and Corollary 4.1 shows that if such solutions were to exist, then (4.25) would be satisfied.

The first assertion in Corollary 4.1 is the known non-existence theorem, see e.g. [9], [13], [27]. The above gives an alternative proof.

4.5 Polynomial Solutions

We first use Theorem 4.1 to give an alternative proof of the known non-existence theorem for polynomial solutions, see e.g. [9], [13], [24].

Corollary 4.2 *There do not exist non-constant polynomial solutions to* (4.1) *for* $n \geq k(k-1)$.

Proof of Corollary 4.2 Assume that there exist non-constant polynomials f_1, $f_2, \dots, f_k$ satisfying (4.1) for some n and k satisfying $n \geq k(k-1)$. Then $k \geq 2$ and V in (4.12) is a polynomial. Hence, $\mathfrak{n}(V) = 0$.

When $n \geq k(k-1)+1$, we apply Theorem 4.1 to polynomial solutions, noting that $\mathfrak{m}(f_j) = \deg f_j$, $j = 1, 2, \dots, k$,

$$\sum_{j=1}^{k} \deg f_j \leq (n - k(k-1)) \sum_{j=1}^{k} \deg f_j \leq k\mathfrak{n}(V) = 0.$$

Then $\sum_{j=1}^{k} \deg f_j = 0$, which means that all of f_1, f_2, $\dots$, f_k are constants, a contradiction.

We next consider the case $n = k(k-1)$. Here we can assume that $k \geq 3$ because it is well known that the equation $f_1^2 + f_2^2 = 1$ cannot possess non-constant polynomial solutions, see e.g. [3], [7], [18]. From Lemma 4.1 (i), $\mathfrak{m}(V) = 0$. Since V is a polynomial, this implies that $\deg V = 0$. Hence, V is a constant.

Since f_1, f_2, $\dots$, f_k are non-constant polynomials, they have zeros. If there exists an f_ℓ, $1 \leq \ell \leq k$, having a zero of multiplicity at least $k-1$, then by Lemma 4.2, V has a zero, a contradiction. Hence, each zero of f_j, $1 \leq j \leq k$ is of multiplicity at most $k-2$. On the other hand, for any polynomial φ, $f_1(\varphi)$, $f_2(\varphi)$, $\dots$, $f_k(\varphi)$ satisfy (4.1). Choosing a suitable polynomial φ, at least one of $f_1(\varphi)$, $f_2(\varphi)$, $\dots$, $f_k(\varphi)$ admits a zero of multiplicity at least $k-1$, which is a contradiction. □

Next we make some observations about the degrees of polynomial solutions of (4.1). In the case when these degrees are all equal, we have the following result, which is a generalization of Lemma 3.5 (i).

Lemma 4.3 *Suppose that* (4.1) *with* $n \geq k-1$ *and* $k \geq 3$ *admits non-constant polynomial solutions* f_1, f_2, $\dots$, f_k *which all have the same degree. If we set* $d = \deg f_j$ *for* $1 \leq j \leq k$ *and set* $v = \deg V$ *where* V *is the function in* (4.12), *then*

$$d\big(k(k-1) - n\big) \geq \frac{k(k-1)}{2} + v + 1. \tag{4.26}$$

Proof of Lemma 4.3 From (4.12),

$$W = V \prod_{j=1}^{k} f_j^{n-k+1}. \tag{4.27}$$

We estimate the degree of both sides of (4.27). The degree of the right-hand side of (4.27) is given by

$$\deg \big(V \prod_{j=1}^{k} f_j^{n-k+1} \big) = k(n-k+1)d + v. \tag{4.28}$$

Let φ be a polynomial. Recall that $U_j(\varphi)$, $1 \leq j \leq k-1$, are homogeneous differential polynomials in φ of degree $k-1$. We note that, for each j, the total derivatives in $U_j(\varphi)$ is equal to j. Hence, $\deg U_j(\varphi)$ is at most $(k-1)\deg \varphi - j$. Since $k \geq 3$, we can use (4.8) and the property of determinants to obtain

$$\deg V_m \leq \sum_{j=1}^{k-1} (d(k-1) - j) - 1 = d(k-1)^2 - \frac{k(k-1)}{2} - 1. \tag{4.29}$$

From (4.7) and (4.29),

$$\deg W = \deg \left(V_m \prod_{\substack{1 \leq j \leq k \\ j \neq m}} f_j^{n-k+1} \right) \leq d(k-1)^2 - \frac{k(k-1)}{2} - 1 + d(k-1)(n-k+1). \tag{4.30}$$

Then (4.26) follows from (4.27), (4.28) and (4.30). □

Various observations from (4.26) can be made. For example, when $k = 4$ and $n = 11$, if there exist non-constant polynomial solutions of (4.1) having the same degree d, then we can assume that $d \geq 7$. When $k = 3$ and $n = 5$, if there exist non-constant polynomial solutions of (4.1) having the same degree d, then we can assume that $d \geq 4$. When $k = 3$ and $n = 2$, we have the following example:

$$(iz^d)^2 + \left(\frac{z^d + 1}{\sqrt{2}} \right)^2 + \left(\frac{z^d - 1}{\sqrt{2}} \right)^2 = 1, \tag{4.31}$$

where d is an arbitrary positive integer. Computing V of (4.31) by (4.27), we obtain $V = 2d^3 z^{3(d-1)}$, which implies $v = \deg V = 3(d-1)$. Hence, the equality in (4.26) holds when $d = 1$ in (4.31).

For the case when $k = 2$, (4.26) does not hold in general. The proof of Lemma 4.3 for this case does not work because V_m is a 1×1 determinant and the property of determinants cannot be used to subtract -1 in (4.29).

For polynomial solutions that do not all have the same degree, we prove the following result for $k = 3$, which is an improvement of Proposition 3.1 (ii).

Proposition 4.1 *Suppose that there exist non-constant polynomials f, g, h that satisfy $f^n + g^n + h^n = 1$, where $2 \leq n \leq 5$, such that $\deg f = \deg g > \deg h$. Then $\deg f = \deg g \geq n$ and $\deg h \geq n - 1$.*

Proof of Proposition 4.1 Since f, g, h are non-constant, the proof is trivial when $n = 2$. Suppose that $3 \leq n \leq 5$. We make the assumption that $\deg h = p$ where $p \leq n - 2$. Since the degree of $1 - h^n$ is equal to np, the degree of $f^n + g^n$ must also be equal to np. We have the factorization

$$f^n + g^n = \prod_{j=1}^{n} (f + c_j g),$$

where $c_1, c_2, \dots, c_n$ are distinct constants satisfying $|c_j| = 1$ for each j. Therefore, it can be deduced that

$$np = \deg(f^n + g^n) \geq (n-1)(p+1) + m$$

for some integer m satisfying $0 \leq m < p + 1$. This gives $p + 1 \geq n + m$. Since it was assumed that $p \leq n - 2$, we obtain the contradiction $m \leq -1$. Hence, we must have $p \geq n - 1$, that is, $\deg h \geq n - 1$. Thus, $\deg f = \deg g \geq n$. □

The following examples show the sharpness of the Proposition 4.1 when $n = 2$ and $n = 3$, respectively:

$$\left(\frac{5 + z^2}{4}\right)^2 + \left(\frac{(3 + z^2)i}{4}\right)^2 + \left(\frac{iz}{2}\right)^2 = 1,$$
$$\left(\frac{1 + z^3}{\sqrt[3]{2}}\right)^3 + \left(\frac{1 - z^3}{\sqrt[3]{2}}\right)^3 + (\omega \sqrt[3]{3} z^2)^3 = 1,$$

where $\omega^3 = -1$, see e.g. [7], [9], [22] regarding the second example.

4.5.1 Examples for The Case of $k = 3$ and $n = 3$

We consider the case $k = 3$ and $n = 3$ of polynomial solutions of (4.1). Namely, target equation is $f^3 + g^3 + h^3 = 1$. Furthermore,we suppose $\deg f = \deg g = \deg h = 3$. As far as we have investigated, we have not found an example of this case. The polynomials f, g, and h are described as follows.

$$f = a_1 z^3 + b_1 z^2 + c_1 z + d_1, \tag{4.32}$$
$$g = a_2 z^3 + b_2 z^2 + c_2 z + d_2, \tag{4.33}$$
$$h = a_3 z^3 + b_3 z^2 + c_3 z + d_3. \tag{4.34}$$

Assume the coefficients as follows and reduce the unknown coefficients.

$$\begin{aligned} a_1 &= 1, d_1 = 0, \\ d_2 &= 0, \\ b_3 &= 0, c_3 = 0, d_3 = 1. \end{aligned}$$

Then the polynomial is denoted as follows.

$$f = z^3 + b_1 z^2 + c_1 z, \tag{4.35}$$
$$g = a_2 z^3 + b_2 z^2 + c_2 z, \tag{4.36}$$
$$h = a_3 z^3 + 1. \tag{4.37}$$

We consider the coefficients of the polynomials f, g, h so as to satisfy the following functional equation.

$$f^3 + g^3 + h^3 = 1. \tag{4.38}$$

However, if a_2, and a_3, are zero, it is not a solution because it is not a three degree polynomial.
We find 54 sets of coefficients by combining equations (4.35) to (4.37). In order to organize the above set of coefficients, we let them as follows.

$$SJ = e^{i\frac{\pi}{9}}, \tag{4.39}$$
$$SM = e^{-i\frac{\pi}{9}}, \tag{4.40}$$
$$WQ = (3(-2 + e^{i\frac{\pi}{3}})^{\frac{1}{9}}, \tag{4.41}$$
$$WN = (3(-2 - e^{i\frac{2\pi}{3}})^{\frac{1}{9}}. \tag{4.42}$$

In addition we use the following relation.

$$SM^j = SJ^{18-j} \; (j = 1, 2, \ldots, 16, 17) \tag{4.43}$$

The 54 sets of solutions are described in the tables in Appendix A using (4.39) to (4.43). We organize the solutions shown in Appendix A into the following general form.

$$\begin{aligned} &a_1 = 1, && a_2 = SJ^{6l-4}, && a_3 = \frac{1}{3} SJ^{6m-3} WN^6, \\ &b_1 = SJ^{6s+2\delta-3} WN, && b_2 = SJ^{6l+6s+2\delta-4} WN, && b_3 = 0, \\ &c_1 = SJ^{-6s+4\delta+15} WN^2, && c_2 = SJ^{6l-6s+4\delta+11} WN^2, && c_3 = 0, \\ &d_1 = 0, && d_2 = 0, && d_3 = 1. \end{aligned}$$

and

$$
\begin{aligned}
&a_1 = 1, && a_2 = SM^{6l-4}, && a_3 = \frac{1}{3}SM^{6m-3}WQ^6,\\
&b_1 = SM^{6s+2\delta-3}WQ, && b_2 = SM^{6l+6s+2\delta-4}WQ, && b_3 = 0,\\
&c_1 = SM^{-6s+4\delta+15}WQ^2, && c_2 = SM^{6l-6s+4\delta+11}WQ^2, && c_3 = 0,\\
&d_1 = 0, && d_2 = 0, && d_3 = 1,
\end{aligned}
$$

where

$$
\begin{aligned}
&l = 1,2,3 \qquad m = 1,2,3 \qquad s = 1,2,3\\
&\delta = 0\ (m=1), \quad \delta = -1\ (m=2), \quad \delta = 1\ (m=3).
\end{aligned}
$$

The above gives an example of polynomial solutions for the case $k = 3$ and $n = 3$, $f^3 + g^3 + h^3 = 1$.

4.6 Four Functions

When $k = 2$, all the values of n for which (4.1) admits non-constant solutions in each of the four function classes (meromorphic, rational, entire, polynomial) have been settled. When $k = 3$ in (4.1), the open questions are as follows: meromorphic ($n = 7, 8$), rational ($n = 6, 7$), entire ($n = 6$), polynomial ($n = 4, 5$). Regarding these statements, see e.g. [7], [8], [9], [13] and the references therein.

Here we discuss the situation when $k = 4$ in (4.1), that is, we address the functional equation

$$f^n + g^n + h^n + w^n = 1. \tag{4.44}$$

For the known non-existence theorems on (4.44) regarding the four function classes, see [9], [13].

There exist transcendental meromorphic solutions of (4.44) when $n = 8$, see [7, Example 3.2]. Equation (4.44) admits non-constant rational solutions when $n = 7$: if $\alpha = e^{2\pi i/3}$, then

$$\frac{1}{63}\left(\frac{1+z^7}{z^2}\right)^7 + \frac{\alpha}{63}\left(\frac{1+\alpha z^7}{z^2}\right)^7 + \frac{\alpha^2}{63}\left(\frac{1+\alpha^2 z^7}{z^2}\right)^7 - (z^3)^7 = 1. \tag{4.45}$$

By replacing z with e^z in (4.45), we obtain transcendental entire solutions of (4.44). Formula (4.45) comes from a general identity [24, p. 486] which shows that (4.44) possesses non-constant rational solutions and transcendental entire solutions for all $n \leq 7$.

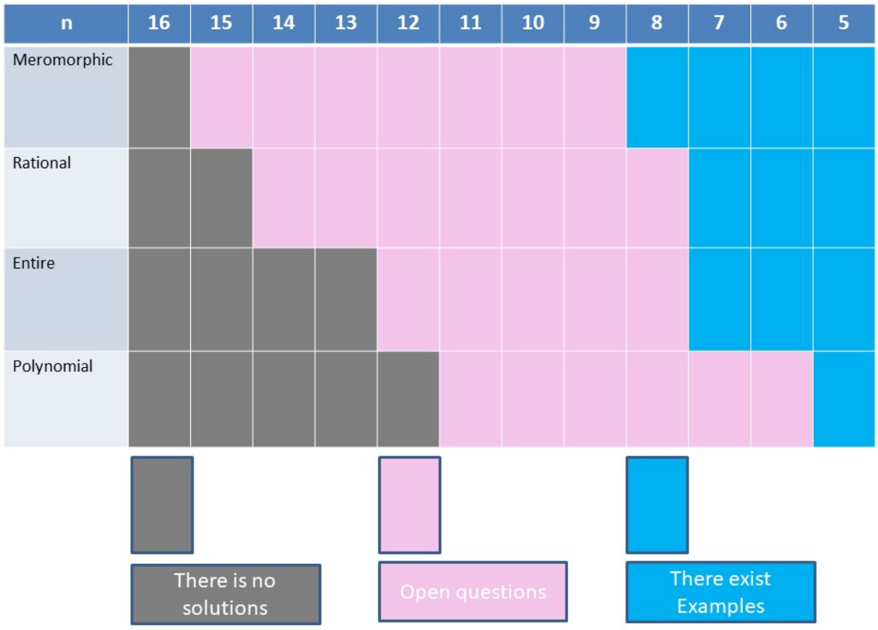
Equation (4.44) admits non-constant polynomial solutions when $n = 5$: if $\alpha = e^{2\pi i/3}$, then

$$\frac{1}{3}(1 + \alpha z^5)^5 + \frac{1}{3}(1 + \alpha^2 z^5)^5 + \frac{1}{3}(1 + z^5)^5 - 10(z^3)^5 = 1. \tag{4.46}$$

Equation (4.46) comes from a general identity [22, p. 50] which shows that (4.44) possesses non-constant polynomial solutions for all $n \leq 5$.

From the above examples and the known non-existence theorems, see e.g. [9], [13], it appears that the open questions on whether or not the equation (4.44) can possess non-constant solutions in each class of functions are as follows: meromorphic functions ($9 \leq n \leq 15$), rational functions ($8 \leq n \leq 14$), entire functions ($8 \leq n \leq 12$) and polynomials ($6 \leq n \leq 11$).

Fig. 4.2: Known Results for $f^n + g^n + h^n + w^n = 1$



Appendix A

Table 4.1: The group of solutions where $a_2 = SJ^2$

a_2	a_3	b_1	b_2	c_1	c_2
SJ^2	$\frac{1}{3}SJ^3WN^6$	SJ^3WN	SJ^8WN	SJ^9WN^2	$SJ^{11}WN^2$
SJ^2	$\frac{1}{3}SJ^3WN^6$	SJ^9WN	$SJ^{14}WN$	SJ^3WN^2	SJ^5WN^2
SJ^2	$\frac{1}{3}SJ^3WN^6$	$SJ^{15}WN$	SJ^2WN	$SJ^{15}WN^2$	$SJ^{17}WN^2$
SJ^2	$\frac{1}{3}SJ^9WN^6$	SJ^1WN	SJ^6WN	SJ^5WN^2	SJ^7WN^2
SJ^2	$\frac{1}{3}SJ^9WN^6$	SJ^7WN	$SJ^{12}WN$	$SJ^{17}WN^2$	SJ^1WN^2
SJ^2	$\frac{1}{3}SJ^9WN^6$	$SJ^{13}WN$	SJ^0WN	$SJ^{11}WN^2$	$SJ^{13}WN^2$
SJ^2	$\frac{1}{3}SJ^{15}WN^6$	SJ^5WN	$SJ^{10}WN$	$SJ^{13}WN^2$	$SJ^{15}WN^2$
SJ^2	$\frac{1}{3}SJ^{15}WN^6$	$SJ^{11}WN$	$SJ^{16}WN$	SJ^7WN^2	SJ^9WN^2
SJ^2	$\frac{1}{3}SJ^{15}WN^6$	$SJ^{17}WN$	SJ^4WN	SJ^1WN^2	SJ^3WN^2

Table 4.2: The group of solutions where $a_2 = SJ^8$

a_2	a_3	b_1	b_2	c_1	c_2
SJ^8	$\frac{1}{3}SJ^3WN^6$	SJ^3WN	$SJ^{14}WN$	SJ^9WN^2	$SJ^{17}WN^2$
SJ^8	$\frac{1}{3}SJ^3WN^6$	SJ^9WN	SJ^2WN	SJ^3WN^2	$SJ^{11}WN^2$
SJ^8	$\frac{1}{3}SJ^3WN^6$	$SJ^{15}WN$	SJ^8WN	$SJ^{15}WN^2$	SJ^5WN^2
SJ^8	$\frac{1}{3}SJ^9WN^6$	SJ^1WN	$SJ^{12}WN$	SJ^5WN^2	$SJ^{13}WN^2$
SJ^8	$\frac{1}{3}SJ^9WN^6$	SJ^7WN	SJ^0WN	$SJ^{17}WN^2$	SJ^7WN^2
SJ^8	$\frac{1}{3}SJ^9WN^6$	$SJ^{13}WN$	SJ^6WN	$SJ^{11}WN^2$	SJ^1WN^2
SJ^8	$\frac{1}{3}SJ^{15}WN^6$	SJ^5WN	$SJ^{16}WN$	$SJ^{13}WN^2$	SJ^3WN^2
SJ^8	$\frac{1}{3}SJ^{15}WN^6$	$SJ^{11}WN$	SJ^4WN	SJ^7WN^2	$SJ^{15}WN^2$
SJ^8	$\frac{1}{3}SJ^{15}WN^6$	$SJ^{17}WN$	$SJ^{10}WN$	SJ^1WN^2	SJ^9WN^2

Table 4.3: The group of solutions where $a_2 = SJ^{14}$

a_2	a_3	b_1	b_2	c_1	c_2
SJ^{14}	$\frac{1}{3}SJ^3WN^6$	SJ^3WN	SJ^2WN	SJ^9WN^2	SJ^5WN^2
SJ^{14}	$\frac{1}{3}SJ^3WN^6$	SJ^9WN	SJ^8WN	SJ^3WN^2	$SJ^{17}WN^2$
SJ^{14}	$\frac{1}{3}SJ^3WN^6$	$SJ^{15}WN$	$SJ^{14}WN$	$SJ^{15}WN^2$	$SJ^{11}WN^2$
SJ^{14}	$\frac{1}{3}SJ^9WN^6$	SJ^1WN	SJ^0WN	SJ^5WN^2	SJ^1WN^2
SJ^{14}	$\frac{1}{3}SJ^9WN^6$	SJ^7WN	SJ^6WN	$SJ^{17}WN^2$	$SJ^{13}WN^2$
SJ^{14}	$\frac{1}{3}SJ^9WN^6$	$SJ^{13}WN$	$SJ^{12}WN$	$SJ^{11}WN^2$	$SJ^{17}WN^2$
SJ^{14}	$\frac{1}{3}SJ^{15}WN^6$	SJ^5WN	SJ^4WN	$SJ^{13}WN^2$	SJ^9WN^2
SJ^{14}	$\frac{1}{3}SJ^{15}WN^6$	$SJ^{11}WN$	$SJ^{10}WN$	SJ^7WN^2	SJ^3WN^2
SJ^{14}	$\frac{1}{3}SJ^{15}WN^6$	$SJ^{17}WN$	$SJ^{16}WN$	SJ^1WN^2	$SJ^{15}WN^2$

Table 4.4: The group of solutions where $a_2 = SM^2$

a_2	a_3	b_1	b_2	c_1	c_2
SM^2	$\frac{1}{3}SM^3WQ^6$	SM^3WQ	SM^8WQ	SM^9WQ^2	$SM^{11}WQ^2$
SM^2	$\frac{1}{3}SM^3WQ^6$	SM^9WQ	$SM^{14}WQ$	SM^3WQ^2	SM^5WQ^2
SM^2	$\frac{1}{3}SM^3WQ^6$	$SM^{15}WQ$	SM^2WQ	$SM^{15}WQ^2$	$SM^{17}WQ^2$
SM^2	$\frac{1}{3}SM^9WQ^6$	SM^1WQ	SM^6WQ	SM^5WQ^2	SM^7WQ^2
SM^2	$\frac{1}{3}SM^9WQ^6$	SM^7WQ	$SM^{12}WQ$	$SM^{17}WQ^2$	SM^1WQ^2
SM^2	$\frac{1}{3}SM^9WQ^6$	$SM^{13}WQ$	SM^0WQ	$SM^{11}WQ^2$	$SM^{13}WQ^2$
SM^2	$\frac{1}{3}SM^{15}WQ^6$	SM^5WQ	$SM^{10}WQ$	$SM^{13}WQ^2$	$SM^{15}WQ^2$
SM^2	$\frac{1}{3}SM^{15}WQ^6$	$SM^{11}WQ$	$SM^{16}WQ$	SM^7WQ^2	SM^9WQ^2
SM^2	$\frac{1}{3}SM^{15}WQ^6$	$SM^{17}WQ$	SM^4WQ	SM^1WQ^2	SM^3WQ^2

Table 4.5: The group of solutions where $a_2 = SM^8$

a_2	a_3	b_1	b_2	c_1	c_2
SM^8	$\frac{1}{3}SM^3WQ^6$	SM^3WQ	$SM^{14}WQ$	SM^9WQ^2	$SM^{17}WQ^2$
SM^8	$\frac{1}{3}SM^3WQ^6$	SM^9WQ	SM^2WQ	SM^3WQ^2	$SM^{11}WQ^2$
SM^8	$\frac{1}{3}SM^3WQ^6$	$SM^{15}WQ$	SM^8WQ	$SM^{15}WQ^2$	SM^5WQ^2
SM^8	$\frac{1}{3}SM^9WQ^6$	SM^1WQ	$SM^{12}WQ$	SM^5WQ^2	$SM^{13}WQ^2$
SM^8	$\frac{1}{3}SM^9WQ^6$	SM^7WQ	SM^0WQ	$SM^{17}WQ^2$	SM^7WQ^2
SM^8	$\frac{1}{3}SM^9WQ^6$	$SM^{13}WQ$	SM^6WQ	$SM^{11}WQ^2$	SM^1WQ^2
SM^8	$\frac{1}{3}SM^{15}WQ^6$	SM^5WQ	$SM^{16}WQ$	$SM^{13}WQ^2$	SM^3WQ^2
SM^8	$\frac{1}{3}SM^{15}WQ^6$	$SM^{11}WQ$	SM^4WQ	SM^7WQ^2	$SM^{15}WQ^2$
SM^8	$\frac{1}{3}SM^{15}WQ^6$	$SM^{17}WQ$	$SM^{10}WQ$	SM^1WQ^2	SM^9WQ^2

Table 4.6: The group of solutions where $a_2 = SM^{14}$

a_2	a_3	b_1	b_2	c_1	c_2
SM^{14}	$\frac{1}{3}SM^3WQ^6$	SM^3WQ	SM^2WQ	SM^9WQ^2	SM^5WQ^2
SM^{14}	$\frac{1}{3}SM^3WQ^6$	SM^9WQ	SM^8WQ	SM^3WQ^2	$SM^{17}WQ^2$
SM^{14}	$\frac{1}{3}SM^3WQ^6$	$SM^{15}WQ$	$SM^{14}WQ$	$SM^{15}WQ^2$	$SM^{11}WQ^2$
SM^{14}	$\frac{1}{3}SM^9WQ^6$	SM^1WQ	SM^0WQ	SM^5WQ^2	SM^1WQ^2
SM^{14}	$\frac{1}{3}SM^9WQ^6$	SM^7WQ	SM^6WQ	$SM^{17}WQ^2$	$SM^{13}WQ^2$
SM^{14}	$\frac{1}{3}SM^9WQ^6$	$SM^{13}WQ$	$SM^{12}WQ$	$SM^{11}WQ^2$	$SM^{17}WQ^2$
SM^{14}	$\frac{1}{3}SM^{15}WQ^6$	SM^5WQ	SM^4WQ	$SM^{13}WQ^2$	SM^9WQ^2
SM^{14}	$\frac{1}{3}SM^{15}WQ^6$	$SM^{11}WQ$	$SM^{10}WQ$	SM^7WQ^2	SM^3WQ^2
SM^{14}	$\frac{1}{3}SM^{15}WQ^6$	$SM^{17}WQ$	$SM^{16}WQ$	SM^1WQ^2	$SM^{15}WQ^2$

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